

MATRIX MODELS AND STOCHASTIC GROWTH IN DONALDSON-THOMAS THEORY

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ABSTRACT. We show that the partition functions which enumerate Donaldson-Thomas invariants of local toric Calabi-Yau threefolds without compact divisors can be expressed in terms of specializations of the Schur measure. We also discuss the relevance of the Hall-Littlewood and Jack measures in the context of BPS state counting and study the partition functions at arbitrary points of the Kähler moduli space. This rewriting in terms of symmetric functions leads to a unitary one-matrix model representation for Donaldson-Thomas theory. We describe explicitly how this result is related to the unitary matrix model description of Chern-Simons gauge theory. This representation is used to show that the generating functions for Donaldson-Thomas invariants are related to tau-functions of the integrable Toda and Toeplitz lattice hierarchies. The matrix model also leads to an interpretation of Donaldson-Thomas theory in terms of non-intersecting paths in the lock-step model of vicious walkers. We further show that these generating functions can be interpreted as normalization constants of a corner growth/last-passage stochastic model.

1. INTRODUCTION AND SUMMARY OF RESULTS

Donaldson-Thomas theory computes enumerative invariants associated to the number of points in the moduli spaces of ideal sheaves with trivial determinant on a three-dimensional Calabi-Yau variety [1, 2]. The partition functions can be rephrased in terms of the counting of noncommutative $U(1)$ instantons in a six-dimensional topological gauge theory [3, 4]. In this way the Donaldson-Thomas partition functions may be regarded as generating functions which count BPS bound states of D0 and D2 branes in a single D6-brane, at least for appropriate values of the B -field. They can also be interpreted combinatorially in terms of the enumeration of plane partitions (three-dimensional Young tableaux) with boundary conditions along the three axes given by ordinary partitions (Young diagrams), where the plane partitions are glued together along common boundaries to form a crystal configuration. This leads to an interpretation of Donaldson-Thomas theory in terms of the statistical mechanics of crystal melting [5]. The melting crystal formulation connects Donaldson-Thomas theory with topological string theory through the formalism of the topological vertex [6]. For local toric backgrounds, the generating functions for Donaldson-Thomas and Gromov-Witten invariants are related by a simple change of variables [7, 8].

In this paper we will construct some new statistical mechanical models of Donaldson-Thomas theory for local toric Calabi-Yau backgrounds which have no compact divisors. We build upon the representation of these partition functions in terms of random Young diagrams, which follows implicitly from the original expansion of the topological vertex [6]. To handle random Young diagrams, one needs to define a probability measure on the set of Young tableaux. The classic example is the Plancherel measure, introduced in the 1970's by Kerov and Vershik [9]. It is of interest in field theory and string theory, wherein the partition functions of $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions [10, 11] and of topological string theory on local toric curves [6] can be conveniently expressed in terms of Plancherel and q -Plancherel measures.

More generally, one can define the Schur measure $\mathcal{M}_{\text{Schur}}$, introduced by Okounkov [12], such that $\mathcal{M}_{\text{Schur}}\{\lambda\}$ for a partition λ is proportional to $\mathfrak{s}_\lambda(x) \mathfrak{s}_\lambda(y)$. Here $\mathfrak{s}_\lambda(x)$ are the Schur polynomials, and x and y are two independent (possibly infinite) sets of variables. There are several important properties satisfied by the Schur measure.¹ For instance, as we shall see in Section 2, after proper specification of the variables it contains the Plancherel and q -Plancherel measures [13, 14] as particular cases. Moreover, the Schur measure has correlation functions of determinantal type, which is a common feature of various problems in statistical mechanics, enumerative combinatorics and probability theory that leads to their explicit solution. One of the remarkable mathematical results of the last decade has been the expression of probability measures on partitions in terms of determinantal point processes [15, 16, 17], which are very often of random matrix type [18, 19]. Some straightforward generalizations of the Schur measure that we shall discuss in Section 2 are defined in terms of the Hall-Littlewood polynomials, or the Jack polynomials which are one-parameter extensions of the Schur polynomials $\mathfrak{s}_\lambda(x)$ [20].²

In this paper our interest in the Schur measure and its generalizations rests in the property that the partition functions of Donaldson-Thomas theory, and also of topological string theory and of the BPS state counting for D6–D2–D0 branes [21], on the backgrounds we consider admit very natural expansions in terms of these measures. More precisely, as we describe in detail in Section 2, they are normalization constants of particular cases of such measures. These special instances come from the specification of the two independent (infinite) sets of parameters x and y in terms of the variables of the generating functions for the Donaldson-Thomas invariants [21]. Several mathematical properties of these measures are utilized throughout this paper. Specific examples of such rewritings in terms of symmetric functions can be found in [22, 23] in the context of the melting crystal model with external potentials, and in [24] in the context of topological string theory (see also [21] for a review).

This representation of the partition functions in terms of symmetric functions is especially interesting due to the well-known combinatorics theorem of Gessel [25], which shows that the normalization constant of the Schur measure can be written as a Toeplitz determinant. By exploiting the classical Heine-Szegö identity which links Toeplitz determinants with unitary matrix models [26], it then leads to a unitary one-matrix model representation for the Donaldson-Thomas partition functions. This is the subject of Section 3. These matrix models appear to be different from the ones previously found for Donaldson-Thomas theory [27] and for topological string theory [28], which involve infinite-dimensional hermitean multi-matrix integrals with non-polynomial potentials. Since our unitary matrix model is characterized by an infinite number of eigenvalues and its weight function is a Jacobi theta-function, we can immediately relate it to the unitary Chern-Simons matrix models [29, 30, 31] with gauge group $U(\infty)$. This is done by completing the unitary matrix model representation of Chern-Simons theory by giving the partition function in terms of a Toeplitz determinant.

The equivalent representations in terms of matrix models, Toeplitz determinants and Schur measures imply, among other things, that the generating functions for Donaldson-Thomas invariants are integrable. We study this problem in detail in Section 4. Using the work of Sato [32] and Segal-Wilson [33], which show that tau-functions of integrable hierarchies admit expansions in terms of Schur functions, we relate the Donaldson-Thomas partition functions to tau-functions of the integrable Toda lattice hierarchy. The equations of the hierarchy, together with the string and divisor equations, uniquely determine the entire theory. We consider various points of view on this issue in gauge theory and string theory which suggest that, in appropriate instances,

¹See [12] for its more representation theoretic properties.

²The Macdonald polynomials [20] are the most general symmetric polynomials known and they include all the other ones as special limiting cases of their two parameters, but we shall not need them here.

the pertinent tau-function is either of 2-Toda or 1-Toda type. For example, in the free fermion formulation [22, 23] there is a hidden symmetry yielding reduction to the 1-Toda lattice hierarchy, which is related to the integrability structure of supersymmetric gauge theory in four and five dimensions. On the other hand, the 1-Toda structure hidden in the 2-Toda formalism is also evident in both ordinary and q -deformed two-dimensional Yang-Mills theory. The results of [34] and [35, §1.4.2] seem to hint that double Hurwitz numbers may be more natural or better suited to describe the branched cover interpretation of these gauge theories, which follows from the philosophy of our treatment as well. This extends the usual description of two-dimensional Yang-Mills theory in terms of simple Hurwitz numbers. Insofar as the q -deformed gauge theory serves as a nonperturbative completion of the A-model topological string theory on certain backgrounds, this integrability structure seems to be related to the fact that equivariant Gromov-Witten theory can be either of 1-Toda or of 2-Toda type. For example, topological string theory on the resolved conifold belongs to the 1-Toda lattice hierarchy, while on toric Fano surfaces it belongs to the 2-Toda lattice hierarchy [36]. The latter geometries contain compact divisors and so fall out of the class of backgrounds that our main line of development applies to in this paper. This explanation for the absence of symmetry reduction of the Toda lattice hierarchy appears to be a generic feature of topological string partition functions.³

A rather generic problem of determinantal type is that of non-intersecting paths or non-intersecting Brownian motion in the continuum. The formalism developed by Karlin and McGregor in the 1950's [39] provided elegant determinantal expressions to describe non-intersecting Brownian motion. These formulas were used in [40, 41] to express Wilson loop observables in Chern-Simons gauge theory as probabilities in a model of N non-intersecting Brownian motion particles, where N is the rank of the gauge group. The system of non-intersecting Brownian motion paths was introduced by de Gennes to study chains of polymers under steric constraints [42]. Later on, Fisher [43, 44] introduced two models of non-intersecting (vicious) random walkers in order to model domain walls in two-dimensional lattice systems: the lock-step model and the random-turns model. The latter model is intimately related to unitary random matrices and the Gross-Witten model [40]. It was also shown in [40, 41] how some of the vicious walker expressions given by Fisher are related to Chern-Simons observables. For example, the probability of reunion for N vicious walkers on a line gives the partition function of Chern-Simons gauge theory on S^3 with gauge group $U(N)$. The problem can be equivalently understood as the Brownian motion of a single particle on the Weyl chamber of the gauge group. In [40], the eventual role of the lock-step model in gauge theory was left as an open question.

The lock-step model of vicious walkers on a one-dimensional lattice allows each walker at the tick of a clock to move either one lattice site to the left or one lattice site to the right, with the restriction that no two walkers may arrive at the same lattice site or pass one another. In Section 5 we will show that our matrix model expressions can be interpreted in terms of non-intersecting paths in this model with infinitely many vicious walkers. This follows directly from [45] which gives a vicious walker interpretation of the generic matrix averages that describe the Donaldson-Thomas partition functions.

In addition, it turns out that the Schur measure employed here is the basis of a generalized version of the corner growth model [18, 19], and hence we can also interpret the Donaldson-Thomas partition functions as normalization constants of this stochastic process. This model is intimately related to other random models, such as the discrete polynuclear growth model,

³This description appears to be equivalent to the topological sigma-model description of Gromov-Witten theory on \mathbb{P}^1 [37], which is equivalent to a large N hermitean one-matrix model with non-polynomial potential that belongs to the 1-Toda lattice hierarchy. In contrast, the equivariant Gromov-Witten theory of \mathbb{P}^1 is governed by the 2-Toda lattice hierarchy [38].

non-intersecting paths, and random tilings of Aztec diamonds [18, 19] (see [15] for a review). It is believed to belong to the Kardar-Parisi-Zhang universality class for stochastic growth processes [46].

These two descriptions imply that Donaldson-Thomas theory has an interpretation in terms of both discrete and continuous random systems. This gives an indication that the non-intersecting paths system of [45] is intimately related to the continuous last passage model in [19]. Our treatment of the BPS bound state partition function holds at all points in the Kähler moduli space, and hence naturally incorporates its discontinuity due to wall-crossing phenomena. In these statistical mechanics models, the jumps across walls of marginal stability have a natural interpretation in terms of the creation or destruction of particles and independent random variables. Furthermore, using results of [45] we shall also see that the Donaldson-Thomas partition functions are generating functions of certain types of infinite integer matrices that satisfy specific symmetry conditions. It would be interesting to relate more directly this enumerative interpretation to the actual integer Donaldson-Thomas invariants that count ideal sheaves (equivalently instantons or D6–D2–D0 bound states).

Note added. While we were completing the present paper, the preprint [47] appeared, whose results overlap with ours, but with a different approach and theme. Their derivation of the unitary matrix models is based on observing that the Chern-Simons matrix model is related to the MacMahon function by expansion of the weight function of the matrix model. Based on the melting crystal formalism, both a free fermion formulation and the Gessel-Viennot determinantal expression which counts non-intersecting paths is then employed to find explicit expressions for the matrix models. As summarised above, our approach is different and yields these same results within a different setting. For example, the non-uniqueness of the weight function of the matrix model, discussed after eq. (4.20) in [47], is emphasised below in (3.18) and the discussion afterwards in a completely different way; this has also been noticed previously for Chern-Simons matrix models. Notice that the Lindström-Gessel-Viennot theorem used in [47] is essentially the same as the Karlin-McGregor theorem that was used previously in [40, 41] to obtain Chern-Simons observables. Although the Karlin-McGregor theorem leads to a two-matrix model, it is explained in [41] how to relate it to the one-matrix model of Chern-Simons gauge theory. This result is also used by Johansson to find the Schur measure (5.5) in the study of the generalized corner growth model that we employ in Section 5.

Acknowledgments. This work was supported in part by grant ST/G000514/1 “String Theory Scotland” from the UK Science and Technology Facilities Council.

2. DONALDSON-THOMAS THEORY AND SYMMETRIC FUNCTION MEASURES ON PARTITIONS

2.1. Affine space. The partition function of Donaldson-Thomas theory in the simplest case of the Calabi-Yau threefold \mathbb{C}^3 is given by the MacMahon function

$$(2.1) \quad Z_{\text{DT}}^{\mathbb{C}^3}(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} \equiv M(q) ,$$

where we choose a string coupling constant g_s such that the quantum parameter $q \equiv e^{-g_s}$ satisfies $|q| < 1$. This is the generating function for plane partitions [48, 49]. They correspond to pointlike instantons, or D0-branes, in the D6-brane gauge theory on \mathbb{C}^3 , with charge equal to the number of boxes in the plane partition. Each plane partition is equivalent to a monomial ideal which corresponds to an ideal sheaf on \mathbb{C}^3 .

As pointed out by [5, 22, 23, 21], this partition function can be rewritten in terms of Schur functions. This expression is the crux of the equivalence between Donaldson-Thomas and Gromov-Witten theories in the toric case [7, 8, 3], with the latter computed from the topological vertex formalism for topological string theory [6] within the melting crystal formulation [5]. However, it also follows directly from the enumerative expression. The Schur polynomials $\mathfrak{s}_\lambda(x)$ [49, 20] constitute a basis of symmetric functions in a given set of variables $x = (x_i)_{i \geq 1}$ and are indexed by Young diagrams (ordinary partitions) $\lambda = (\lambda_i)_{i \geq 1}$, with $\lambda_i \geq \lambda_{i+1} \geq 0$ giving the length of the i -th row. If the variables x are regarded as eigenvalues of some matrix $M \in sl_n$, then $\mathfrak{s}_\lambda(x) \equiv \text{Tr}_\lambda(M)$ is the trace of M in the irreducible sl_n -representation associated to λ . The Schur polynomials may also be more explicitly defined in terms of the skew-symmetric polynomials $\mathfrak{a}_\mu = \det_{i,j}(x_i^{\mu_j + n - j})$ as $\mathfrak{s}_\lambda(x) \equiv \mathfrak{a}_\lambda(x)/\mathfrak{a}_0(x)$. By using the Cauchy identity [49]

$$(2.2) \quad \sum_{\lambda} \mathfrak{s}_\lambda(x) \mathfrak{s}_\lambda(y) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} \equiv \mathcal{Z} ,$$

and considering the case of an infinite number of variables with the specializations $x_i = q^{i-1/2}$ and $y_j = q^{j-1/2}$, then (2.2) directly gives the expression [21] for the Donaldson-Thomas partition function of \mathbb{C}^3 in terms of Schur functions,

$$(2.3) \quad Z_{\text{DT}}^{\mathbb{C}^3}(q) = \sum_{\lambda} \mathfrak{s}_\lambda(q^{i-1/2})^2 .$$

Let us phrase this result in terms of the Schur measure introduced by Okounkov in [12]. It assigns to each partition λ the weight $\mathcal{M}_{\text{Schur}}\{\lambda\} = \frac{1}{\mathcal{Z}} \mathfrak{s}_\lambda(x) \mathfrak{s}_\lambda(y)$, where $\mathfrak{s}_\lambda(x)$ are Schur functions. Then

$$(2.4) \quad \mathcal{P}_N\{\lambda\} = \frac{1}{\mathcal{Z}} \sum_{\lambda: \lambda_1 \leq N} \mathfrak{s}_\lambda(x) \mathfrak{s}_\lambda(y)$$

is the probability that the number of boxes in the first row of the associated Young diagram is $\leq N$. Taking the limit $N \rightarrow \infty$, the normalization constant \mathcal{Z} is given by (2.2). Thus the Donaldson-Thomas partition function (2.3) is given by the normalization constant of the Schur measure, specialized at $x_i = q^{i-1/2}$ and $y_j = q^{j-1/2}$.

By making the specification $\mathfrak{s}_\lambda(1, 1, \dots) = \dim \lambda$, the Schur measure contains the Plancherel measure

$$(2.5) \quad \mathcal{M}_{\text{Planch}}\{\lambda\} = \frac{(\dim \lambda)^2}{|\lambda|!} ,$$

where the dimension of the corresponding irreducible representation of the symmetric group $S_{|\lambda|}$, with $|\lambda| := \sum_i \lambda_i$ the weight of the representation, is given by

$$(2.6) \quad \dim \lambda = \prod_{u \in \lambda} \frac{1}{h(u)}$$

Here $h(u) \equiv \lambda_i + \lambda'_j - i - j + 1$ is the hook length of the box $u = (i, j)$ in λ , and λ' denotes the conjugate partition to λ .⁴ Our specification is instead $q^{-|\lambda|/2} \mathfrak{s}_\lambda(1, q, \dots, q^{n-1})$, and we know that [49, 20]

$$(2.7) \quad \mathfrak{s}_\lambda(1, q, \dots, q^{n-1}) = q^{n(\lambda)} \prod_{u \in \lambda} \frac{[n + c(u)]}{[h(u)]} ,$$

⁴The conjugate of a partition is obtained by interchanging rows and columns in its Ferrers graph.

where $n(\lambda) \equiv \sum_{i \geq 1} (i-1) \lambda_i$ and, for each box $u = (i, j)$ of the diagram λ , $c(u) \equiv j - i$ is the content of u . The square brackets here denote q -numbers,

$$(2.8) \quad [a] \equiv \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}} ,$$

and the right-hand side of (2.7) is the q -deformation of the dimensions of sl_n representations, i.e. the quantum dimensions $\dim_q \lambda$ [50]. Then the Schur measure with this specialization is a q -Plancherel measure, essentially the q -deformed Plancherel measure discussed in [13].⁵ Thus the normalization constant of the q -Plancherel measure gives the Donaldson-Thomas partition function on \mathbb{C}^3 .

2.2. Conifold. There are two non-isomorphic crepant resolutions of the conifold singularity $z_1 z_2 - z_3 z_4 = 0$ in \mathbb{C}^4 [51]. The topological string partition function of the resolved conifold $\mathbb{X} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ can be expressed in terms of the Donaldson-Thomas generating function

$$(2.9) \quad Z_{\text{DT}}^{\mathbb{X}}(q, Q) = M(-q)^2 M(Q, -q)^{-1}$$

where $Q = e^{-t}$ with t the Kähler modulus of the base $\mathbb{P}^1 \hookrightarrow \mathbb{X}$ such that $|Q| < 1$, and

$$(2.10) \quad M(Q, q) = \prod_{n=1}^{\infty} \frac{1}{(1 - Q q^n)^n}$$

is the generalized MacMahon function, the generating function for weighted plane partitions. The extra parameter Q weights the contributions from “fractional instantons” stuck at the resolution of the conifold singularity, or D2-branes in the D6-brane gauge theory on \mathbb{X} , with charge equal to the number of diagonal boxes of the plane partition. On the other hand, the Donaldson-Thomas theory of the noncommutative crepant resolution of the conifold singularity enumerates framed cyclic representations of a quiver with the Klebanov-Witten superpotential [51], or equivalently of the corresponding quotient path algebra \mathbb{A} . The corresponding partition function can be computed by counting partitions of a two-coloured pyramid of length one,⁶ associated with a perfect matching of a brane tiling, and is given by [51]

$$(2.11) \quad Z_{\text{DT}}^{\mathbb{A}}(q, Q) = M(-q)^2 M(Q, -q)^{-1} M(Q^{-1}, -q)^{-1} .$$

The appropriate measure in this case involves the Hall-Littlewood polynomials $\mathfrak{p}_\lambda(x; v)$ [20], one of two generalizations of the Schur polynomials. They are symmetric polynomials in x , homogeneous of degree $|\lambda|$, and with coefficients in $\mathbb{Z}[v]$. As with the Schur polynomials, the Hall-Littlewood polynomials can be extended to Hall-Littlewood functions involving an infinite number of variables. They are defined by

$$(2.12) \quad \mathfrak{p}_\lambda(x; v) = \sum_{\sigma \in S_n / S_n^\lambda} \sigma \left(x \prod_{\lambda_i > \lambda_j} \frac{x_i - v x_j}{x_i - x_j} \right) ,$$

where S_n^λ is the subgroup of the symmetric group S_n consisting of permutations that leave $\lambda = (\lambda_1, \dots, \lambda_n)$ invariant, and $\sigma(f(x)) = f(\sigma(x))$ with $\sigma(x) = x_1^{\sigma_1} \cdots x_n^{\sigma_n}$. The parameter v serves to interpolate between the Schur polynomials $\mathfrak{s}_\lambda(x)$ at $v = 0$ and the monomial symmetric functions $\mathfrak{m}_\lambda(x) = \sum_{\sigma \in S_n / S_n^\lambda} \sigma(x)$ at $v = 1$. The corresponding Cauchy identity for Hall-Littlewood polynomials is

$$(2.13) \quad \sum_{\lambda} b_\lambda(v) \mathfrak{p}_\lambda(x; v) \mathfrak{p}_\lambda(y; v) = \prod_{i, j \geq 1} \frac{1 - v x_i y_j}{1 - x_i y_j} \equiv \mathcal{Z}_{\text{HL}} ,$$

⁵Other q -deformations are introduced in [14], where only one of the Schur functions is q -specialized.

⁶The two colours are weighted by the variables $q_0 = q/Q$ and $q_1 = Q$.

where

$$(2.14) \quad b_\lambda(v) = \prod_{i=1}^{\lambda_1} (v)_{m_i(\lambda)} .$$

Here $(v)_0 \equiv 1$, $(v)_m \equiv \prod_{1 \leq j \leq m} (1 - v^j)$ for $m \in \mathbb{N}$, and $(v)_\infty \equiv \prod_{j \in \mathbb{N}} (1 - v^j)$, while $m_i(\lambda)$ is the number of parts of λ equal to i .

In parallel to the \mathbb{C}^3 case above, if we make the specification $x_i = (-q)^{i-1/2}$ and $y_j = (-q)^{j-1/2}$ in (2.13), and set the parameter v equal to either Q or Q^{-1} , then we obtain expressions for the conifold Donaldson-Thomas partition functions analogous to (2.3) given by

$$(2.15) \quad Z_{\text{DT}}^{\times}(q, Q) = M(-q) \sum_{\lambda} b_\lambda(Q) \mathfrak{p}_\lambda((-q)^{i-1/2}; Q)^2$$

and

$$(2.16) \quad Z_{\text{DT}}^{\text{A}}(q, Q) = \sum_{\lambda} b_\lambda(Q) \mathfrak{p}_\lambda((-q)^{i-1/2}; Q)^2 \sum_{\mu} b_\mu(Q^{-1}) \mathfrak{p}_\mu((-q)^{i-1/2}; Q^{-1})^2 .$$

Likewise, one can define a Hall-Littlewood measure on partitions given by

$$(2.17) \quad \mathcal{M}_{\text{HL}}\{\lambda\} = \frac{1}{Z_{\text{HL}}} b_\lambda(v) \mathfrak{p}_\lambda(x; v) \mathfrak{p}_\lambda(y; v) ,$$

with the normalization Z_{HL} given by (2.13). This leads to the Donaldson-Thomas partition functions on the conifold (2.15) and (2.16), when specializing to $x_i = (-q)^{i-1/2}$ and $y_j = (-q)^{j-1/2}$. This specialized measure can be regarded as a Hall-Littlewood deformation of the q -Plancherel measure.

We have seen that the generalization of symmetric functions provided by the Hall-Littlewood polynomials is appropriate to the Donaldson-Thomas partition functions of the conifold, in contrast to the simpler case of \mathbb{C}^3 . This is especially true in the noncommutative setting (2.16), where even the two MacMahon function factors in (2.11) are automatically included. The appearance of Hall-Littlewood polynomials in topological string theory has already been discussed in [24]. In particular, the expression (2.15) for the partition function of the resolved conifold is given there. However, the reduced partition function

$$(2.18) \quad \tilde{Z}_{\text{DT}}^{\times}(q, Q) = \frac{Z_{\text{DT}}^{\times}(q, Q)}{M(-q)^2} = M(Q, -q)^{-1}$$

factors out the contributions from the degree zero subschemes (regular D0-branes) of \mathbb{X} , and it can be written solely in terms of Schur functions by using the dual Cauchy identity

$$(2.19) \quad \sum_{\lambda} \mathfrak{s}_\lambda(x) \mathfrak{s}_{\lambda'}(y) = \prod_{i,j \geq 1} (1 + x_i y_j)$$

together with the scaling property $\mathfrak{s}_\lambda(Qx) = Q^{|\lambda|} \mathfrak{s}_\lambda(x)$ to write

$$(2.20) \quad \tilde{Z}_{\text{DT}}^{\times}(q, Q) = \sum_{\lambda} (-Q)^{|\lambda|} \mathfrak{s}_\lambda((-q)^{i-1/2}) \mathfrak{s}_{\lambda'}((-q)^{i-1/2}) .$$

Similarly, one has

$$(2.21) \quad \tilde{Z}_{\text{DT}}^{\text{A}}(q, Q) = \sum_{\lambda, \mu} (-Q)^{|\lambda| - |\mu|} \mathfrak{s}_\lambda((-q)^{i-1/2}) \mathfrak{s}_{\lambda'}((-q)^{i-1/2}) \mathfrak{s}_\mu((-q)^{i-1/2}) \mathfrak{s}_{\mu'}((-q)^{i-1/2}) .$$

2.3. Orbifold. Consider the action of the cyclic group \mathbb{Z}_2 on \mathbb{C}^3 generated by $(z_1, z_2, z_3) \mapsto (-z_1, -z_2, z_3)$. The crepant resolution of the orbifold singularity $\mathbb{C}^3/\mathbb{Z}_2$ given by the \mathbb{Z}_2 -Hilbert scheme is $\mathbb{Y} = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(0) = \mathcal{O}_{\mathbb{P}^1}(-2) \times \mathbb{C}$, where the first factor is the minimal (Hirzebruch-Jung) resolution of the A_1 Klein singularity $\mathbb{C}^2/\mathbb{Z}_2$ [21]. The corresponding Donaldson-Thomas partition function is similar to that of the conifold (2.9) and reads [52]

$$(2.22) \quad Z_{\text{DT}}^{\mathbb{Y}}(q, Q) = M(-q)^2 M(Q, -q) ,$$

where again $Q = e^{-t}$. The Donaldson-Thomas invariants of the noncommutative crepant resolution, given by the quiver algebra of the McKay quiver associated to the A_1 singularity [21], are the same as the orbifold Donaldson-Thomas invariants of the quotient stack $[\mathbb{C}^3/\mathbb{Z}_2]$ [52, 53]. They are labelled by \mathbb{Z}_2 -representations ρ and enumerate \mathbb{Z}_2 -invariant zero-dimensional subschemes $Y \subset \mathbb{C}^3$ with $H^0(\mathcal{O}_Y) = \rho$. The corresponding partition function can be computed by counting configurations of two-coloured boxes, with the colours corresponding to the two irreducible representations of \mathbb{Z}_2 , and is given by [52]

$$(2.23) \quad Z_{\text{DT}}^{\mathbb{C}^3/\mathbb{Z}_2}(q, Q) = M(-q)^2 M(Q, -q) M(Q^{-1}, -q) .$$

The reduced orbifold partition functions are “dual” to those of the conifold, in the sense that they are related through

$$(2.24) \quad \tilde{Z}_{\text{DT}}^{\mathbb{Y}}(q, Q) = \tilde{Z}_{\text{DT}}^{\mathbb{X}}(q, Q)^{-1} \quad \text{and} \quad \tilde{Z}_{\text{DT}}^{\mathbb{C}^3/\mathbb{Z}_2}(q, Q) = \tilde{Z}_{\text{DT}}^{\mathbb{A}}(q, Q)^{-1} .$$

Hence one can regard the orbifold generating functions in terms of Hall-Littlewood measures by using the formulas of the preceding subsection. Alternatively, they can also be written solely in terms of Schur functions as

$$(2.25) \quad \tilde{Z}_{\text{DT}}^{\mathbb{Y}}(q, Q) = \sum_{\lambda} Q^{|\lambda|} \mathfrak{s}_{\lambda}((-q)^{i-1/2})^2$$

and

$$(2.26) \quad \tilde{Z}_{\text{DT}}^{\mathbb{C}^3/\mathbb{Z}_2}(q, Q) = \sum_{\lambda, \mu} Q^{|\lambda| - |\mu|} \mathfrak{s}_{\lambda}((-q)^{i-1/2})^2 \mathfrak{s}_{\mu}((-q)^{i-1/2})^2 .$$

2.4. BPS state counting. For the toric Calabi-Yau backgrounds X of interest in this paper, which have no compact divisors, it was shown in [54, 55] that the closed topological string partition function can be expressed in terms of genus zero Gopakumar-Vafa invariants as

$$(2.27) \quad Z_{\text{top}}^X(q, Q) = M(q)^{\chi(X)/2} \prod_{\beta \in H_2(X, \mathbb{Z})^+} \prod_m M(q^m Q^{\beta}, q)^{-N_m^{\beta}}$$

where $\chi(X)$ is the topological Euler characteristic of X , the integer N_m^{β} counts the number of BPS states of M2-branes in curve class $\beta = (\beta_1, \dots, \beta_s) \in H_2(X, \mathbb{Z})$ and with intrinsic $SU(2)$ spin m , and $Q^{\beta} = Q_1^{\beta_1} \dots Q_s^{\beta_s}$. This expression describes the generating functions Z_{BPS}^X , counting BPS bound states of a single D6 brane wrapping X with D2 and D0 branes, in different chambers of the Kähler moduli space which are separated by walls of marginal stability where BPS states decay or form. This includes the chambers that lead to the Donaldson-Thomas partition functions at both commutative and noncommutative points in the Kähler moduli space [51, 56, 57]. In any chamber, the BPS partition function is given by

$$(2.28) \quad Z_{\text{BPS}}^X = Z_{\text{top}}^X(q, Q) Z_{\text{top}}^X(q, Q^{-1}) .$$

The symmetric function expansion in this case involves the other generalization of the Schur polynomials, the Jack polynomials $j_{\lambda}^{(\alpha)}(x)$ [58, 20] with $j_{\lambda}^{(\alpha=1)}(x) = \mathfrak{s}_{\lambda}(x)$. They satisfy a Cauchy

identity

$$(2.29) \quad \sum_{\lambda} c_{\lambda}(\alpha) j_{\lambda}^{(\alpha)}(x) j_{\lambda}^{(\alpha)}(y) = \prod_{i,j \geq 1} (1 - x_i y_j)^{-1/\alpha} ,$$

where $c_{\lambda}(\alpha)$ are rational functions of the parameter α which have been calculated in [58]. Due to the power $1/\alpha$ in (2.29), they are well suited to describe the topological string partition function as written in (2.27). The generic expression (2.27) can thus be written in terms of the Jack polynomials as

$$(2.30) \quad \frac{Z_{\text{top}}^X(q, Q)}{M(q)^{\chi(X)/2}} = \prod_{\beta \in H_2(X, \mathbb{Z})^+} \prod_m \sum_{\lambda} c_{\lambda}(-1/N_m^{\beta}) q^{m|\lambda|} Q^{|\lambda|\beta} j_{\lambda}^{(-1/N_m^{\beta})}(q^{i-1/2})^2 ,$$

and the powers of the MacMahon function can be expressed as

$$(2.31) \quad M(q)^{\chi(X)/2} = \sum_{\lambda} c_{\lambda}(2/\chi(X)) j_{\lambda}^{(2/\chi(X))}(q^{i-1/2})^2 .$$

For the backgrounds considered in this paper, the (unreduced) noncommutative partition functions are related to those of X by the wall-crossing factor $W^X = \tilde{Z}_{\text{DT}}^X(q, Q^{-1})$, which describes the crossing of an infinite number of walls in going from the noncommutative point to the large volume point [51, 56]. We can index the walls crossed by $i \in \mathbb{N}$ and factorize the wall-crossing factor as $W^X = \prod_{i \geq 1} W_i^X$, with W_i^X the jump of the BPS state partition function across the i -th wall. This yields the partition function (2.28) in the n -th chamber

$$(2.32) \quad Z_{\text{BPS}}^X(n) = \mathcal{W}_n^X Z_{\text{DT}}^X(q, Q) ,$$

where $\mathcal{W}_n^X = \prod_{i \geq n} W_i^X$.

For the resolved conifold $X = \mathbb{X}$, this prescription yields $W_i^{\mathbb{X}} = (1 - (-q)^i)^i$ and

$$(2.33) \quad \mathcal{W}_n^{\mathbb{X}} = ((-q)^n Q^{-1}; -q)_{\infty}^n \tilde{Z}_{\text{DT}}^{\mathbb{X}}(q, (-q)^n Q^{-1})$$

where $(a; q)_{\infty} \equiv \prod_{j \in \mathbb{N}} (1 - a q^j)$. The expansion of (2.32) in Hall-Littlewood functions is given by

$$(2.34) \quad \begin{aligned} Z_{\text{BPS}}^{\mathbb{X}}(n) &= ((-q)^n Q^{-1}; -q)_{\infty}^n \sum_{\lambda} b_{\lambda}(Q) \mathfrak{p}_{\lambda}((-q)^{i-1/2}; Q)^2 \\ &\quad \times \sum_{\mu} b_{\mu}((-q)^n Q^{-1}) \mathfrak{p}_{\mu}((-q)^{i-1/2}; (-q)^n Q^{-1})^2 . \end{aligned}$$

This function interpolates between $Z_{\text{BPS}}^{\mathbb{X}}(1) = Z_{\text{DT}}^{\mathbb{A}}$ (after reparametrization) and $Z_{\text{BPS}}^{\mathbb{X}}(\infty) = Z_{\text{DT}}^{\mathbb{X}}$, at the noncommutative and large volume limit points, through values of n corresponding to chambers in the Kähler cone of \mathbb{X} . For the resolution of the A_1 quotient singularity $X = \mathbb{Y}$, using duality one has $\mathcal{W}_n^{\mathbb{Y}} = (\mathcal{W}_n^{\mathbb{X}})^{-1}$. The expansion of the BPS state partition function (2.32) in Schur functions is then given by

$$(2.35) \quad \begin{aligned} Z_{\text{BPS}}^{\mathbb{Y}}(n) &= ((-q)^n Q^{-1}; -q)_{\infty}^n \\ &\quad \times \sum_{\lambda, \mu, \nu} Q^{|\mu| - |\nu|} (-q)^{n|\nu|} \mathfrak{s}_{\lambda}((-q)^{i-1/2})^2 \mathfrak{s}_{\mu}((-q)^{i-1/2})^2 \mathfrak{s}_{\nu}((-q)^{i-1/2})^2 . \end{aligned}$$

2.5. Generalizations. These constructions generalize in a natural way to all local toric Calabi-Yau threefolds without compact divisors. Let us briefly illustrate this through a representative set of examples, generalizing the conifolds and \mathbb{Z}_2 -orbifolds considered above. Let G be a finite subgroup of $SU(2)$ acting on \mathbb{C}^3 via the natural embedding $SU(2) \subset SU(3)$. The crepant resolution of the orbifold \mathbb{C}^3/G given by the G -Hilbert scheme of \mathbb{C}^3 is $\mathbb{Y}_G = \mathbb{S}_G \times \mathbb{C}$, where \mathbb{S}_G is the minimal ADE resolution of the double point singularity \mathbb{C}^2/G . By the McKay correspondence [21], the non-trivial irreducible G -representations correspond to simple roots of an

associated ADE root system Δ , the collection of which gives a basis for $H_2(\mathbb{Y}_G, \mathbb{Z})$. Let $\Delta^+ \subset \Delta$ be the set of positive roots, and n the number of irreducible representations of G . Then the Donaldson-Thomas partition function generalizes (2.22) to [59]

$$(2.36) \quad Z_{\text{DT}}^{\mathbb{Y}_G}(q, Q) = M(-q)^n \prod_{\beta \in \Delta^+} M(Q^\beta, -q) .$$

The noncommutative Donaldson-Thomas invariants arising from the McKay quiver associated to the affine ADE Dynkin diagram are the same as the orbifold Donaldson-Thomas invariants of \mathbb{C}^3/G [53, 59]. The corresponding partition function $Z_{\text{DT}}^{\mathbb{C}^3/G}(q, Q)$ is given by the same formula (2.36) but with the product now ranging over the full root lattice Δ [59]. Hence the corresponding reduced partition functions can be expanded in Schur functions as

$$(2.37) \quad \tilde{Z}_{\text{DT}}^{\mathbb{Y}_G}(q, Q) = \prod_{\beta \in \Delta^+} \sum_{\lambda} Q^{|\lambda|} \mathfrak{s}_\lambda((-q)^{i-1/2})^2 ,$$

and the identical formula for $\tilde{Z}_{\text{DT}}^{\mathbb{C}^3/G}(q, Q)$ involving a product over all roots. The corresponding BPS state partition functions in the various chambers can be worked out as before.

For the special case $G = \mathbb{Z}_n$, the products run over roots of the A_{n-1} Lie algebra and the partition function (2.36) specializes to [52, 51]

$$(2.38) \quad Z_{\text{DT}}^{\mathbb{Y}_{\mathbb{Z}_n}}(q, Q) = M(-q)^n \prod_{1 \leq i \leq j < n} M(Q_{[i,j]}, -q) ,$$

where $Q_{[i,j]} \equiv Q_i Q_{i+1} \cdots Q_j$. This is (after reparametrization) the generating function for n -coloured plane partitions. It is “dual” to the Donaldson-Thomas partition function for the generalized conifold geometry \mathbb{X}_n [60, 61, 54] which is given by

$$(2.39) \quad Z_{\text{DT}}^{\mathbb{X}_n}(q, Q) = M(-q)^n \prod_{1 \leq i \leq j < n} M(Q_{[i,j]}, -q)^{N_{ij}} ,$$

where $N_{ij} = -(-1)^{n_{ij}}$ with n_{ij} the number of internal edges between vertices i and j in the toric web diagram for \mathbb{X}_n . The corresponding reduced function is expanded in Schur functions as

$$(2.40) \quad \begin{aligned} \tilde{Z}_{\text{DT}}^{\mathbb{X}_n}(q, Q) &= \prod_{\substack{1 \leq i \leq j < n \\ N_{ij} = +1}} \sum_{\lambda} (Q_{[i,j]})^{|\lambda|} \mathfrak{s}_\lambda((-q)^{k-1/2})^2 \\ &\times \prod_{\substack{1 \leq i \leq j < n \\ N_{ij} = -1}} \sum_{\mu} (-Q_{[i,j]})^{|\mu|} \mathfrak{s}_\mu((-q)^{k-1/2}) \mathfrak{s}_{\mu'}((-q)^{k-1/2}) . \end{aligned}$$

These formulas also formally apply to the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and its symmetric resolution, the closed topological vertex geometry [52]. They can also be extended to some non-toric geometries without compact divisors, such as the formal toric Calabi-Yau threefolds considered in [62, 63] (see also [54]). The resulting expressions are straightforward but somewhat tedious to write down, and one must be somewhat more careful with the convergence of the various infinite products involved. We omit the details.

3. UNITARY MATRIX MODELS FOR DONALDSON-THOMAS THEORY

3.1. Chern-Simons theory and Toeplitz determinants. The interest of Toeplitz determinants in physics begins with Onsager’s study of the two-dimensional Ising model, as he succeeded

in showing that the diagonal spin correlation function is given by an $N \times N$ Toeplitz determinant [64, 65]. Another classic example is the one-plaquette model of two-dimensional Yang-Mills theory with gauge group $U(N)$, whose partition function is given by [66]

$$(3.1) \quad Z_N(\lambda) = \det_{1 \leq i, j \leq N} [I_{i-j}(2\lambda)] ,$$

where $I_n(z)$ is the modified Bessel function order n . The same problem was shortly afterwards studied using unitary one-matrix models [67], leading to the well-known Gross-Witten matrix model

$$(3.2) \quad Z_N(\lambda) = \prod_{j=1}^N \int_0^{2\pi} \frac{d\theta_j}{2\pi} \exp\left(\lambda(e^{i\theta_j} + e^{-i\theta_j})\right) \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^2 .$$

In recent work [31] we showed that the unitary matrix model for $U(N)$ Chern-Simons gauge theory on the three-sphere S^3 is, in a certain sense, a q -deformation of the Gross-Witten model. Here we shall show that its partition function can also be represented as a Toeplitz determinant.

The reason for the existence of two equivalent representations is a direct relationship between Toeplitz determinants and unitary random matrix models, found long before the introduction of matrix models. It is given by the Heine-Szegö identity [26]

$$(3.3) \quad \prod_{j=1}^N \int_0^{2\pi} \frac{d\theta_j}{2\pi} f(e^{i\theta_j}) \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^2 = D_N(f) ,$$

where $D_N(f)$ is the $N \times N$ Toeplitz determinant with symbol f ,

$$(3.4) \quad D_N(f) = \det_{1 \leq j, k \leq N} [\widehat{f}(j-k)] ,$$

with $\widehat{f}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ir\theta}) d\theta$, $r \in \mathbb{Z}$ the Fourier coefficients of the symbol function. The symbol function $f(z)$ is the weight function of the corresponding matrix model; we assume throughout that it lives in $C^\infty(S^1)$. In the case of the Gross-Witten model, one uses the identity

$$(3.5) \quad \exp\left(\lambda(z + z^{-1})\right) = \sum_{n=-\infty}^{\infty} I_n(2\lambda) z^n$$

to establish the equivalence of the two expressions (3.1) and (3.2).

Let us now compute the $U(N)$ Chern-Simons free energy on S^3 in terms of a Toeplitz determinant. The unitary matrix model is given by the partition function [29]

$$(3.6) \quad Z_{\text{CS}}^{U(N)}(S^3) \equiv \prod_{j=1}^N \int_0^{2\pi} \frac{d\theta_j}{2\pi} \Theta(e^{i\theta_j}|q) \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^2$$

with

$$(3.7) \quad \Theta(z|q) = \sum_{j=-\infty}^{\infty} q^{j^2/2} z^j .$$

We have seen that the entries of the Toeplitz determinant (3.4) are given by the Fourier coefficients of the symbol, which is the weight function of the unitary matrix model. In this case, the symbol is the theta-function $\Theta(z|q)$ and the entries of the matrix are automatically at our disposal. Then the determinant is

$$(3.8) \quad D_N(\Theta) = \begin{vmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_{-1} & a_0 & \cdots & a_{N-2} \\ \vdots & \vdots & & \vdots \\ a_{-N+1} & a_{-N+2} & \cdots & a_0 \end{vmatrix} \quad \text{with} \quad a_j = q^{j^2/2} .$$

Hence the required determinant is $\det_{i,j} [q^{-(i-j)^2/2}]$, which is known to give rise to the Chern-Simons partition function [40]

$$(3.9) \quad Z_{\text{CS}}^{U(N)}(S^3) = \det_{1 \leq i, j \leq N} [q^{-(i-j)^2/2}] = \prod_{k=1}^{N-1} (1 - q^k)^{N-k} .$$

Notice that the non-trivial part of the determinant, $\det_{i,j} [q^{ij}]$, is just a Vandermonde determinant.

A more general Toeplitz determinant, with symbol

$$(3.10) \quad \sum_{j=0}^{\infty} q^{j^2/2} z^j$$

was already computed in [68]. It was proven there, by induction, that the determinant (3.8) with coefficient a_m instead of a_0 is given by

$$(3.11) \quad D_{m,N} = (q^{m(m-1)/2})^N \prod_{k=1}^{N-1} (1 - q^k)^{N-k} .$$

They consider the case $m > N$ because their symbol is (3.10), but if the symbol is the full theta-function (3.7) we can take $m = 0$ and recover $D_{0,N} = Z_{\text{CS}}^{U(N)}(S^3)$ given in (3.9).

Of course, knowledge of the Toeplitz determinant result directly implies, due to (3.3), that there is a unitary matrix model representation with a theta-function as weight function, a model found by other means in [29]. In [30] it was shown that the inverse of the theta-function also leads to a viable Chern-Simons matrix model. We shall see that there is an equivalence between these results and the ensuing representation of Donaldson-Thomas partition functions in terms of unitary matrix models.

3.2. Donaldson-Thomas theory on \mathbb{C}^3 . As in the case of Chern-Simons gauge theory on S^3 , we can write the partition function of Donaldson-Thomas theory on \mathbb{C}^3 as a Toeplitz determinant. For this, we use Gessel's formula for the product of Schur polynomials in terms of a Toeplitz determinant, which in terms of the Schur measure reads [25]

$$(3.12) \quad \mathcal{P}_N(x, y) \equiv \sum_{\lambda: \lambda_1 \leq N} \mathfrak{s}_\lambda(x) \mathfrak{s}_\lambda(y) = D_N(A_{i-j}) ,$$

where

$$(3.13) \quad A_i = A_i(x, y) = \sum_{l=0}^{\infty} \mathfrak{h}_{l+i}(x) \mathfrak{h}_l(y) ,$$

and $\mathfrak{h}_r(x) = \sum_i x_i^r$ is the r -th complete symmetric function. Then the Donaldson-Thomas partition function is given by

$$(3.14) \quad Z_{\text{DT}}^{\mathbb{C}^3}(q) = \lim_{N \rightarrow \infty} \mathcal{P}_N(x_i = q^{i-1/2}, y_i = q^{i-1/2}) .$$

It is convenient to consider the symbol of the Toeplitz determinant [25, 69]

$$(3.15) \quad f_{\mathbb{C}^3}(z) = \sum_{i=-\infty}^{\infty} A_i(x, y) z^i = \prod_{j \geq 1} (1 - y_j z^{-1})^{-1} (1 - x_j z)^{-1} .$$

Taking into account the Heine-Szegö identity (3.3), the Donaldson-Thomas partition function can then be written as the $N \rightarrow \infty$ limit of an $N \times N$ matrix model

$$(3.16) \quad Z_{\text{DT}}^{\mathbb{C}^3}(q) = \lim_{N \rightarrow \infty} \prod_{n=1}^N \int_0^{2\pi} \frac{d\theta_n}{2\pi} \prod_{j=1}^{\infty} \left(1 - q^{j-1/2} e^{-i\theta_n}\right)^{-1} \left(1 - q^{j-1/2} e^{i\theta_n}\right)^{-1} \times \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^2 .$$

Using the Jacobi triple product formula for the theta-function

$$(3.17) \quad \Theta(z|q) = \prod_{j=1}^{\infty} (1 - q^j) \left(1 + q^{j-1/2} z^{-1}\right) \left(1 + q^{j-1/2} z\right) ,$$

we can write

$$(3.18) \quad Z_{\text{DT}}^{\mathbb{C}^3}(q) = A_{\infty} \prod_{n=1}^{\infty} \int_0^{2\pi} \frac{d\theta_n}{2\pi} \frac{1}{\Theta(-e^{i\theta_n}|q)} \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^2 ,$$

where $A_{\infty} = \lim_{N \rightarrow \infty} (q)_{\infty}^{-N}$. Thus the Donaldson-Thomas partition function is a unitary $N = \infty$ one-matrix model with weight function $w(\theta) = \Theta(-e^{i\theta}|q)^{-1}$.

Recall that the matrix model for $U(N)$ Chern-Simons gauge theory on S^3 is a unitary matrix model with weight function $w'(\theta) = \Theta(e^{i\theta}|q)$ and N eigenvalues [29, 31]. We relate the two models and the corresponding partition functions in a more precise way below. Before doing that, we can also demonstrate the relationship using an equivalent matrix model representation of the Donaldson-Thomas partition function, based again on the Cauchy identity but now written as

$$(3.19) \quad \sum_{\lambda} \mathfrak{s}_{\lambda'}(x) \mathfrak{s}_{\lambda'}(y) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}$$

where λ' are the conjugate (transposed) partitions. The symbol in this case is [69]

$$(3.20) \quad f'_{\mathbb{C}^3}(z) = \prod_{j \geq 1} (1 + y_j z^{-1}) (1 + x_j z) ,$$

and the corresponding matrix model representation is

$$(3.21) \quad Z_{\text{DT}}^{\mathbb{C}^3}(q) = \lim_{N \rightarrow \infty} \prod_{n=1}^N \int_0^{2\pi} \frac{d\theta_n}{2\pi} \prod_{j=1}^{\infty} \left(1 + q^{j-1/2} e^{-i\theta_n}\right) \left(1 + q^{j-1/2} e^{i\theta_n}\right) \times \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^2 .$$

Using (3.17) again, the expression in terms of theta-functions is then

$$(3.22) \quad Z_{\text{DT}}^{\mathbb{C}^3}(q) = C_{\infty} \prod_{n=1}^{\infty} \int_0^{2\pi} \frac{d\theta_n}{2\pi} \Theta(e^{i\theta_n}|q) \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^2 ,$$

with $C_{\infty} = \lim_{N \rightarrow \infty} (q)_{\infty}^{-N}$. The matrix model representation of the MacMahon function [48] has not appeared before.

3.3. Chern-Simons/Donaldson-Thomas correspondence. The two matrix model representations, (3.18) and (3.22), parallels the situation in Chern-Simons theory, since the weight function $w(\theta) = \Theta(-e^{i\theta}|q)^{-1}$ can also be used for a Chern-Simons matrix model. The Chern-Simons unitary matrix model representation thus not only matches (3.22) but also (3.18). This result appeared in [30, eq. (4.8)], where the non-uniqueness of the matrix model representation was pointed out, based on the undetermined moment problem for the Stieltjes-Wigert weight function. It follows from the explicit expression, given by Askey, for a weight function that

has the same moments as the log-normal distribution, which is given by the inverse of a theta-function [70]

$$(3.23) \quad w_\gamma(z) = c_\gamma \frac{z^{\gamma-1}}{(-z; q)_\infty (-q/z; q)_\infty} .$$

If we set $z = \sqrt{q} e^{i\theta}$ and choose $\gamma = 0$, then by the triple product formula (3.17) we directly obtain the Chern-Simons — or finite N — matrix model version of (3.18).⁷

Thus in both cases, we see that the right-hand side is essentially the matrix model for $U(N)$ Chern-Simons gauge theory on S^3 when $N = \infty$. Indeed, considering the corresponding generating functions, we have

$$(3.24) \quad Z_{\text{CS}}^{U(N)}(S^3) = \prod_{j=1}^{N-1} \frac{(1-q^j)^N}{(1-q^j)^j} \quad \text{and} \quad Z_{\text{DT}}^{\mathbb{C}^3}(q) = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)^j} .$$

Then (3.22) implies $Z_{\text{DT}}^{\mathbb{C}^3}(q) = C_\infty \lim_{N \rightarrow \infty} Z_{\text{CS}}^{U(N)}(S^3)$, which also follows from (3.24) and

$$(3.25) \quad (q)_\infty^{-N} = \prod_{j=1}^{\infty} (1-q^j)^{-N} .$$

From the very definitions of their partition functions, the Donaldson-Thomas free energy $F_{\text{DT}}^{\mathbb{C}^3} = \log Z_{\text{DT}}^{\mathbb{C}^3}$ and the Chern-Simons free energy $F_{\text{CS}}^{U(N)} = \log Z_{\text{CS}}^{U(N)}$ at $N = \infty$ thus satisfy the simple relationship

$$(3.26) \quad F_{\text{DT}}^{\mathbb{C}^3}(q) = F_{\text{CS}}^{U(\infty)}(S^3) - \lim_{N \rightarrow \infty} N \sum_{j=1}^{\infty} \log(1-q^j) .$$

Hence, via a precise infinite renormalization, the Donaldson-Thomas free energy is exactly the free energy of $U(N)$ Chern-Simons gauge theory in the limit $N \rightarrow \infty$.

The factor $\lim_{N \rightarrow \infty} (q)_\infty^{-N}$ that links the two partition functions has a natural interpretation. As we have seen, the q -Pochhammer symbol

$$(3.27) \quad (q)_n = \prod_{j=1}^n (1-q^j)$$

is the building block of the Chern-Simons partition function. In the limit $n \rightarrow \infty$, it also appears as the partition function of $U(N)$ topologically twisted Vafa-Witten $\mathcal{N} = 4$ gauge theory [71] on \mathbb{C}^2 , which is given by [21]

$$(3.28) \quad Z_{U(N)}^{\mathbb{C}^2}(q) = (q)_\infty^{-N} .$$

We can thus write

$$(3.29) \quad Z_{\text{DT}}^{\mathbb{C}^3}(q) = \lim_{N \rightarrow \infty} Z_{U(N)}^{\mathbb{C}^2}(q) Z_{\text{CS}}^{U(N)}(S^3) .$$

It would be interesting to better understand this relationship from the point of view of the six-dimensional $U(1)$ topological gauge theory underlying the Donaldson-Thomas invariants [3, 4]. If we regard \mathbb{C}^3 as the trivial line bundle over \mathbb{C}^2 , then this result suggests that the fibre degrees of freedom in the six-dimensional gauge theory can be integrated out, leaving its natural gauge theory counterpart on \mathbb{C}^2 and on the boundary S^3 , at infinite rank. This sort of reduction of the gauge theory partition function is demonstrated in the case of local toric surfaces in [72].

⁷In [30], the particular case $\gamma = -3/2$ was considered, which gives one of the most common normalizations of the Stieltjes-Wigert polynomials.

However, in (3.28) the (complexified) gauge coupling parameter is $q = \exp(2\pi i \tau)$ with $\tau = \frac{4\pi i}{g_{\text{YM}}^2}$ (for vanishing θ -angle). If we use the relationship with the topological string coupling constant $g_s = g_{\text{YM}}^2/2$, then the quantum parameter is $q = \exp(-4\pi^2/g_s)$, while the quantum parameter in Chern-Simons theory is $q = \exp(-g_s)$. Thus in (3.29) the Chern-Simons partition function should be understood in its dual form after performing a Gauss summation. This is the correct setting for merging the four-dimensional Vafa-Witten theory with its corresponding boundary Chern-Simons gauge theory [73, 72]. Indeed, the equality (3.29) can be regarded as the large N limit of the partition function for q -deformed Yang-Mills theory on the two-sphere S^2 [73], which describes the relationship between the two gauge theories. We will discuss the relation between Donaldson-Thomas theory and q -deformed gauge theories in the next section.

3.4. Conifold Donaldson-Thomas theory. The case of the conifold partition functions can be similarly studied. Using $\mathfrak{s}_\lambda(Qx) = Q^{|\lambda|} \mathfrak{s}_\lambda(x)$, the symbol associated to the slightly more general expansion (2.20) is given by

$$(3.30) \quad f_{\mathbb{X}}(z) = \prod_{j \geq 1} (1 - Q y_j z^{-1})^{-1} (1 + x_j z) .$$

Hence the Donaldson-Thomas partition function of the resolved conifold (without degree zero contributions) has the matrix model representation

$$(3.31) \quad \tilde{Z}_{\text{DT}}^{\mathbb{X}}(q, Q) = \lim_{N \rightarrow \infty} \prod_{n=1}^N \int_0^{2\pi} \frac{d\theta_n}{2\pi} \prod_{j=1}^{\infty} \frac{1 + q^{j-1/2} e^{i\theta_n}}{1 - Q q^{j-1/2} e^{-i\theta_n}} \prod_{k < l} |e^{i\theta_k} - e^{i\theta_l}|^2 .$$

The noncommutative conifold partition function (2.21) can be written as the product of two Toeplitz determinants $D(f) = \det T(f)$, where $T(f) = [\hat{f}(j-k)]_{j,k \geq 1}$ is the corresponding $\ell^2(\mathbb{N})$ Toeplitz operator. However, the product of two Toeplitz operators is not of Toeplitz form [74]. To deal with this problem, one considers all Toeplitz matrices $T_N(f) \equiv \Pi_N T(f) \Pi_N$ for $N \in \mathbb{N}$ together, where Π_N is the orthogonal projection onto the Fourier modes $1, \dots, N$. Then a sequence of products of Toeplitz matrices $\{T_N(f) T_N(g)\}_{N \in \mathbb{N}}$ is asymptotically equivalent to the sequence of Toeplitz matrices $\{T_N(fg)\}_{N \in \mathbb{N}}$ [74]. The requisite condition is that the Fourier coefficients of the symbol $t_k = \hat{f}(k)$ (the entries of the Toeplitz matrix) are absolutely summable

$$(3.32) \quad \sum_{k=-\infty}^{\infty} |t_k| < \infty .$$

A sequence of Toeplitz matrices $T_N = [t_{j-k}]_{1 \leq j,k \leq N}$ for which the t_k are absolutely summable is said to be in the Wiener class. Likewise, a function $f(z)$ defined on the circle S^1 is said to be in the Wiener class if it has a Fourier series expansion with absolutely summable Fourier coefficients. If the symbols are of Wiener class, then one has the convergence result [74]

$$(3.33) \quad \lim_{N \rightarrow \infty} \|T_N(f) T_N(g) - T_N(fg)\| = 0$$

where $\| - \|$ denotes the operator norm on finite-dimensional matrices.

In our case, the symbols are given by theta-functions which have Fourier coefficients of the type $t_k = q^{k^2/2}$ with $0 < q < 1$. Absolute summability is consequently satisfied and hence the symbols are of Wiener class. The limit $N \rightarrow \infty$ is precisely the one that we are studying, and

hence we can write

$$(3.34) \quad \begin{aligned} \tilde{Z}_{\text{DT}}^{\text{A}}(q, Q) &= \lim_{N \rightarrow \infty} \prod_{n=1}^N \int_0^{2\pi} \frac{d\theta_n}{2\pi} \prod_{j=1}^{\infty} \frac{(1 + q^{j-1/2} e^{i\theta_n})^2}{(1 - Q q^{j-1/2} e^{-i\theta_n}) (1 - Q^{-1} q^{j-1/2} e^{-i\theta_n})} \\ &\quad \times \prod_{k < l} |e^{i\theta_k} - e^{i\theta_l}|^2, \end{aligned}$$

with the limit understood in the sense of norm convergence.

3.5. Orbifold Donaldson-Thomas theory. The orbifold partition functions are treated analogously. The symbol associated to the Schur function expansion (2.25) is

$$(3.35) \quad f_{\text{Y}}(z) = \prod_{j \geq 1} (1 - Q y_j z^{-1})^{-1} (1 - x_j z)^{-1}.$$

This leads to the matrix model representation

$$(3.36) \quad \begin{aligned} \tilde{Z}_{\text{DT}}^{\text{Y}}(q, Q) &= \lim_{N \rightarrow \infty} \prod_{n=1}^N \int_0^{2\pi} \frac{d\theta_n}{2\pi} \prod_{j=1}^{\infty} \frac{1}{(1 - Q q^{j-1/2} e^{-i\theta_n}) (1 - q^{j-1/2} e^{i\theta_n})} \\ &\quad \times \prod_{k < l} |e^{i\theta_k} - e^{i\theta_l}|^2. \end{aligned}$$

Likewise, by using the same arguments which led to (3.34), the noncommutative orbifold partition function (2.26) admits the matrix model representation

$$(3.37) \quad \begin{aligned} \tilde{Z}_{\text{DT}}^{\text{C}^3/\mathbb{Z}_2}(q, Q) &= \lim_{N \rightarrow \infty} \prod_{n=1}^N \int_0^{2\pi} \frac{d\theta_n}{2\pi} \prod_{j=1}^{\infty} \frac{(1 - q^{j-1/2} e^{i\theta_n})^{-2}}{(1 - Q q^{j-1/2} e^{-i\theta_n}) (1 - Q^{-1} q^{j-1/2} e^{-i\theta_n})} \\ &\quad \times \prod_{k < l} |e^{i\theta_k} - e^{i\theta_l}|^2. \end{aligned}$$

As pointed out in [74], the arguments which led to (3.34) easily extend to the more general case of products of $m > 2$ Toeplitz matrices. In particular, one has the convergence results

$$(3.38) \quad \lim_{N \rightarrow \infty} \|T_N(f_1) \cdots T_N(f_m) - T_N(f_1 \cdots f_m)\| = 0.$$

This enables us to write down matrix model representations for all generalizations considered in the previous section. For example, the BPS state partition function (2.35) can be expressed as

$$(3.39) \quad \begin{aligned} Z_{\text{BPS}}^{\text{Y}}(n) &= ((-q)^n Q^{-1}; -q)_{\infty}^n C_{\infty} \lim_{N \rightarrow \infty} \prod_{m=1}^N \int_0^{2\pi} \frac{d\theta_m}{2\pi} \Theta(e^{i\theta_m} | q) \\ &\quad \times \prod_{j=1}^{\infty} \frac{(1 - q^{j-1/2} e^{i\theta_m})^{-2}}{(1 - Q q^{j-1/2} e^{-i\theta_m}) (1 - Q^{-1} (-q)^n q^{j-1/2} e^{-i\theta_m})} \prod_{k < l} |e^{i\theta_k} - e^{i\theta_l}|^2. \end{aligned}$$

Similarly, the generalized orbifold partition function (2.37) is

$$(3.40) \quad \tilde{Z}_{\text{DT}}^{\text{Y}_G}(q, Q) = \lim_{N \rightarrow \infty} \prod_{n=1}^N \int_0^{2\pi} \frac{d\theta_n}{2\pi} \prod_{j=1}^{\infty} \prod_{\beta \in \Delta^+} \frac{(1 - q^{j-1/2} e^{i\theta_n})^{-1}}{1 - Q^{\beta} q^{j-1/2} e^{-i\theta_n}} \prod_{k < l} |e^{i\theta_k} - e^{i\theta_l}|^2,$$

while the generalized conifold partition function (2.39) can be expressed as

$$\begin{aligned}
 \tilde{Z}_{\text{DT}}^{\times n}(q, Q) &= \lim_{N \rightarrow \infty} \prod_{m=1}^N \int_0^{2\pi} \frac{d\theta_m}{2\pi} \prod_{p=1}^{\infty} \prod_{\substack{1 \leq i \leq j < n \\ N_{ij} = +1}} \frac{(1 - q^{p-1/2} e^{i\theta_m})^{-1}}{1 - Q_{[i,j]} q^{p-1/2} e^{-i\theta_m}} \\
 (3.41) \quad &\times \prod_{\substack{1 \leq i \leq j < n \\ N_{ij} = -1}} \frac{1 + q^{p-1/2} e^{i\theta_m}}{1 - Q_{[i,j]} q^{p-1/2} e^{-i\theta_m}} \prod_{k < l} |e^{i\theta_k} - e^{i\theta_l}|^2,
 \end{aligned}$$

and so on.

The Toeplitz determinant can also be expressed as a Fredholm determinant [75, 76], which suggests a deep underlying integrability structure, as we discuss in detail in the next section. This applies directly to the Chern-Simons partition function, which is given by a finite $N \times N$ Toeplitz determinant, for which the result of [76] immediately applies. This result does not strictly apply in the limit $N \rightarrow \infty$ which yields the Donaldson-Thomas partition function. In this limit, the Fredholm determinant representation converges to 1 and the infinite Toeplitz determinant is given by the normalization constant of the Schur measure [76]. This agrees with our computation, wherein the matrix model representation of Donaldson-Thomas theory follows directly from the Toeplitz determinant representation of the Cauchy identity (3.12) and the Heine-Szegö identity (3.3). Proper specialization of the symmetric functions then yields the Donaldson-Thomas partition functions. For the first part of this procedure, one can alternatively use the results of [77], where the Cauchy identity is expressed directly as a generic matrix model average. Other instances where the partition function can be written as a Fredholm determinant are the partition function of two-dimensional quantum gravity [78] and the grand canonical partition function of $c = 1$ string theory with vortex excitations [79].

4. INTEGRABILITY STRUCTURE

4.1. Toda and Toeplitz lattice hierarchies. The expansions in Schur functions are also useful to establish a relationship between Donaldson-Thomas theory and the theory of integrable hierarchies, a connection that is generally expected whenever a matrix model formulation is available [80, 81]. Using the results of [82], it is straightforward to identify the Donaldson-Thomas partition functions as particular instances of a tau-function of the 2-Toda lattice hierarchy [83]. More precisely, they are given by tau-functions of the Toeplitz lattice hierarchy [82, 84], which is a reduction of the 2-Toda lattice hierarchy.⁸ The tau-functions of the 2-Toda lattice hierarchy $\tau_n(t, s)$, $n \in \mathbb{Z}$ depend on two sets of time variables $t, s \in \mathbb{C}^\infty$ and are defined by the Hirota bilinear equations. They can also be written as the determinant of a semi-infinite moment matrix

$$(4.1) \quad \tau_n(t, s) = \det m_n(t, s) .$$

In the case of the reduction to the Toeplitz lattice hierarchy, this moment matrix is a Toeplitz matrix (whereas in the reduction to the standard Toda lattice hierarchy it is a Hankel matrix), and the components of the vector $\tau(t, s) = (\tau_0(t, s) = 1, \tau_1(t, s), \dots)$ of tau-functions of the Toeplitz lattice hierarchy satisfy [82]

$$(4.2) \quad \tau_n(t, s) = \sum_{\lambda: \lambda_1 \leq n} \mathfrak{s}_\lambda(t) \mathfrak{s}_\lambda(-s) .$$

⁸As pointed out in [82], the Toeplitz lattice hierarchy is better known as the Ablowitz-Ladik hierarchy [85] which arises in discretizations of the non-linear Schrödinger equation, but we follow the terminology of [82, 84]. The other reduction of the 2-Toda lattice hierarchy leads to the standard Toda lattice hierarchy [82]. The Toeplitz reduction (on S^1) and the Toda reduction (on \mathbb{R}) are essentially equivalent [86].

Hence if the times are taken to be $t_i = q^{i-1/2}$ and $s_j = -q^{j-1/2}$, then

$$(4.3) \quad \tau_\infty(q^{i-1/2}, -q^{j-1/2}) = Z_{\text{DT}}^{\text{C}^3}(q) .$$

On the other hand, if the two sets of time variables are taken to be $t_i = \sqrt{Q} q^{i-1/2}$ and $s_j = -\sqrt{Q} q^{j-1/2}$, then

$$(4.4) \quad \tau_\infty(\sqrt{Q} q^{i-1/2}, -\sqrt{Q} q^{j-1/2}) = \tilde{Z}_{\text{DT}}^{\text{Y}}(q, Q) = \tilde{Z}_{\text{DT}}^{\text{X}}(q, Q)^{-1} .$$

Thus while one begins within the formalism of the 2-Toda lattice hierarchy, the Donaldson-Thomas partition functions are a particular case of a reduction of the 2-Toda lattice hierarchy. This reduction is equivalent to the one-dimensional Toda lattice hierarchy, a result that has also been shown in the context of the melting crystal picture [23]. While our partition functions involve two sets of Schur polynomials, the two sets of times are taken to be equal. That this reduces the system to a one-dimensional Toda hierarchy is shown in [23]. Using the free fermionic representation of a deformation of the expansion (2.25), they show that it is a 2-Toda tau-function that satisfies

$$(4.5) \quad \tau(t, s) = \tau(t - s, 0) ,$$

and thus reduces to a one-dimensional Toda lattice hierarchy.

More generally, the tau-function of the two-component KP hierarchy is also a function of two sets of time variables and admits the double Schur function expansion [32]

$$(4.6) \quad \tau_{\text{KP}}(t, s) = \sum_{\lambda, \mu} c_{\lambda\mu} \mathfrak{s}_\lambda(t) \mathfrak{s}_\mu(-s) ,$$

where $c_{\lambda\mu}$ are Plücker coordinates of a point on a two-component analog of the Sato grassmannian [32] and have well-known determinant expressions. (The Schur expansion for tau-functions of the ordinary KP hierarchy consists of only one set of Schur functions.) The case of the 2-Toda lattice hierarchy follows from this one, as it only has an additional discrete index n as in (4.2). More precisely, following [87] these tau-functions can be written as images of the Plücker map corresponding to projection along a basis element of the charge N sector of a fermionic Fock space as

$$(4.7) \quad \tau_{N,g}^{(2)}(t, s) = \sum_{\lambda, \mu} B_{N,g}(\lambda, \mu) \mathfrak{s}_\lambda(t) \mathfrak{s}_\lambda(-s) .$$

Here $B_{N,g}(\lambda, \mu)$ are Plücker coordinates of the image of an element $g \in Gr_{\mathcal{H}_+^N}(\mathcal{H})$ in the grassmannian of subspaces of the Hilbert space $\mathcal{H} = L^2(S^1)$ whose orthogonal projections onto the subspace $\mathcal{H}_+ \subset \mathcal{H}$, consisting of functions that admit holomorphic extensions to the interior of $S^1 \subset \mathbb{C}$, have Fredholm index N . Then the noncommutative Donaldson-Thomas partition function $\tilde{Z}_{\text{DT}}^{\text{A}}(q, Q)$ is a particular case of (4.7) with unit Plücker coordinates.

The particular case of (4.7) for the identity element $g = e$ is [87]

$$(4.8) \quad \tau_{N,e}^{(2)}(t, s) = \sum_{\lambda} \mathfrak{s}_\lambda(t) \mathfrak{s}_\lambda(-s)$$

which, with the specification of the time variables given above, describes the partition functions $Z_{\text{DT}}^{\text{C}^3}(q)$ and $\tilde{Z}_{\text{DT}}^{\text{Y}}(q, Q)$, and reduces to a one-dimensional Toda lattice tau-function. With the matrix model expression for $\tilde{Z}_{\text{DT}}^{\text{A}}(q, Q)$, one can interpret it as the product of two one-dimensional Toda tau-functions. The more general partition functions described earlier are likewise given as products of 1-Toda tau-functions.

4.2. Isomonodromic tau-functions. Following [88], a somewhat more conjectural relationship with tau-functions and integrable systems can also be established, using the fact that the Donaldson-Thomas and Chern-Simons partition functions are Toeplitz determinants. The result of [88] shows that some Toeplitz determinants can be interpreted as certain isomonodromic tau-functions, which coincide with those of Jimbo, Miwa and Ueno [89]. They focus on a particular case of the Gessel identity (3.12) where one set of variables $x = (x_1, \dots, x_n)$ is generic and left unspecified, while the other set has the specification $y = (1, \dots, 1)$. In this case the associated symbol is

$$(4.9) \quad f_{\text{JMU}}(z) = e^{1/z} \prod_{j=1}^n (1 + x_j z) .$$

However, as pointed out in [88], their derivation of integrable partial differential equations can be applied to any Toeplitz determinant for which the logarithmic derivative of the associated symbol $f(z)$ is a rational function of z .

The Chern-Simons partition functions for gauge group $U(N)$ have finite-dimensional Toeplitz determinant representation and symbol given by a theta-function (3.17) after the proper specification, i.e. $n \rightarrow \infty$. This symbol violates the rational behaviour required of its logarithmic derivative, unless we truncate the products at some large value $M < n$. The extension to infinite support of the variables $x = (x_i)$ is left as an open problem in [88]. In this “finite-dimensional” approximation, the calculation of the Toeplitz determinant in Section 3.1 proceeds based on the Fourier coefficients up to a_{M-1} , and corresponds to a truncation of the symbol to index M . Then the truncated Chern-Simons partition functions can be identified with the Jimbo-Miwa-Ueno tau-functions corresponding to the (generalized) Schlesinger isomonodromy deformation equations of the 2×2 linear matrix ordinary differential equations which have M simple poles in \mathbb{C} and one irregular singular point at infinity of Poincaré index one.

The rigorous proof that the Chern-Simons partition function is an isomonodromic tau-function is beyond the scope of the present paper.⁹ It is then an interesting open problem to show that the Donaldson-Thomas partition functions are tau-functions of a similar system of non-linear partial differential equations, which have both infinitely many poles in the complex plane and are given by infinite-dimensional Toeplitz determinants. The isomonodromic tau-functions are known [90] to be intimately related to the KP tau-functions discussed above and introduced in [32], hence it should be possible to further establish a stronger relationship between these two families of integrable hierarchies and Donaldson-Thomas theory.

4.3. Nekrasov functions. The expansion of Donaldson-Thomas partition functions in terms of the q -Plancherel measure is also useful for establishing a relationship with $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions with Casimir operators, since Nekrasov’s formulas [91] can be also written in terms of the Plancherel measure [10, 11]. This sort of relationship has already been noticed, within the crystal melting picture in [22, 23] and also in [92] within a slightly different context. In particular, for the noncommutative $U(1)$ gauge theory¹⁰ the instanton partition function $Z_{\text{inst}}(a, t, \epsilon_1, \epsilon_2)$ can be expressed as a sum over partitions [10, 11]. In the

⁹This proof presumably follows the arguments indicated in [28] that the essential characteristics are independent of the cutoff M on the length of the representation associated to λ .

¹⁰In general, one considers an abelian $\mathcal{N} = 2$ gauge theory with instantons. The noncommutative gauge theory is one possibility, whereby the field theory is embedded in the Ω -background with toric deformation parameters (ϵ_1, ϵ_2) . Other possibilities are the gauge theories of fractional D3-branes at an ADE singularity or of the D5/NS5 brane system wrapping a \mathbb{P}^1 in $K3$ [11].

Calabi-Yau case when $\epsilon_1 = -\epsilon_2 = \hbar$, one has

$$(4.10) \quad Z_{\text{inst}}(a, t, \hbar) = \sum_{\lambda} \frac{(\dim \lambda)^2}{(-\hbar^2)^{|\lambda|}} \exp\left(-\frac{1}{\hbar^2} \sum_{k \geq 1} t_k \frac{\text{ch}_{k+1}(a, \lambda)}{k+1}\right),$$

where $t = (t_k)$ are coupling constants, $\text{ch}_{k+1}(a, \lambda)$ are the Chern characters of the partition λ , and a is the vacuum expectation value of the vector multiplets. Thus it coincides with the principal specification¹¹ of the Schur functions

$$(4.11) \quad \dim \lambda = \lim_{q \rightarrow 1} \lim_{N \rightarrow \infty} \mathfrak{s}_{\lambda}(1, q, \dots, q^{N-1}).$$

As in [10, 11], let us consider the case $t_1 \neq 0$ and $t_2 = t_3 = \dots = 0$, i.e. all higher Casimir operators are turned off. Using the Chern character $\text{ch}_2(a, \lambda) = a^2 + 2\hbar^2 |\lambda|$, in that case we have

$$(4.12) \quad Z_{\text{inst}}(a, t_1, \hbar) = \sum_{\lambda} \frac{\mathfrak{s}_{\lambda}(1, 1, \dots)^2}{(-\hbar^2)^{|\lambda|}} \exp\left(-\frac{t_1}{\hbar^2} a^2 - t_1 |\lambda|\right).$$

Consider now the $\mathbb{C}^3/\mathbb{Z}_2$ orbifold Donaldson-Thomas partition function (2.25), and take its $q \rightarrow 1$ limit using

$$(4.13) \quad \lim_{q \rightarrow 1} \mathfrak{s}_{\lambda}(q^{i-1/2})^2 = g_s^{-2|\lambda|} \mathfrak{s}_{\lambda}(1, 1, \dots)^2$$

where $q = e^{-g_s}$. Then the partition function $\tilde{Z}_{\text{DT}}^{\text{Y}}(-q, Q)$ in the limit $q \rightarrow 1$ is the non-equivariant limit of Nekrasov's partition function with $Q = e^{-t_1}$ and $g_s = i\hbar \rightarrow 0$,

$$(4.14) \quad \lim_{q \rightarrow 1} \tilde{Z}_{\text{DT}}^{\text{Y}}(-q = -e^{-i\hbar}, Q = e^{-t_1}) = e^{t_1 a^2 / 2\hbar^2} Z_{\text{inst}}(a, t, \hbar).$$

The equivariant case which is directly related with Donaldson-Thomas theory is that of the five-dimensional $\mathcal{N} = 1$ supersymmetric $U(1)$ gauge theory (or K-theory version of the original gauge theory), because its instanton expansion involves the q -Plancherel measure instead of the Plancherel measure above [22, 23].¹² Its partition function coincides with $\tilde{Z}_{\text{DT}}^{\text{Y}}(q, Q)$ and $\tilde{Z}_{\text{DT}}^{\text{X}}(q, Q)^{-1}$ given in (2.25) [22]. As we have seen above and as is also shown explicitly in [23], this partition function is a tau-function of the one-dimensional Toda lattice hierarchy and not of the two-dimensional Toda lattice hierarchy.

4.4. Hurwitz theory. Another gauge theory partition function that can be immediately written in terms of the Plancherel measure is the heat kernel expansion of two-dimensional Yang-Mills theory [93, 94]. On the sphere $S^2 \cong \mathbb{P}^1$, from this expansion or from the matrix model representation [95] the partition function can be written in terms of Schur polynomials as

$$(4.15) \quad Z_{\text{YM}}^{U(N)}(\mathbb{P}^1) = \sum_{\lambda} \mathfrak{s}_{\lambda}(1, \dots, 1)^2 q^{C_2(\lambda)},$$

where $\mathfrak{s}_{\lambda}(1, \dots, 1) = \dim \lambda$ is the dimension of the irreducible representation of the $U(N)$ gauge group corresponding to $\lambda = (\lambda_1, \dots, \lambda_N)$, and $C_2(\lambda)$ is the quadratic Casimir invariant of the representation. Hence in the large N limit, whereby the gauge group is $U(\infty)$, it coincides with the Nekrasov partition function (4.10) with only the first two Casimir operators turned on, an observation already made by [10] (see also [92]).

The large N limit picks out the chiral sector of the two-dimensional gauge theory which receives contributions from only “small” representations λ to the partition function (4.15),

¹¹Principal means that it utilizes Schur functions (infinite number of variables), as in Donaldson-Thomas theory.

¹²The string coupling g_s here is then interpreted as the circumference of the compactified fifth dimension.

weighted by the linear Casimir operator. Then the $U(\infty)$ chiral partition function is related to the Nekrasov function (4.12), and one has

$$(4.16) \quad Z_{\text{YM}}^{U(\infty)}(\mathbb{P}^1)^+ = \lim_{q \rightarrow 1} \tilde{Z}_{\text{DT}}^{\text{Y}}(-q, Q).$$

Of course, the q -deformation of this theory is more intimately related to Donaldson-Thomas theory, as then no limit is required. We shall discuss this case in more detail below. We first point out that the expansion of the partition function in terms of Schur polynomials allows us to apply results of Okounkov [34] and relate the partition functions with tau-functions of the Toda lattice hierarchy, while it further enables the study of their role as generating functions for ramified coverings of \mathbb{P}^1 [35].

Let us begin by briefly summarizing the Toda lattice results of [34]. Let $H_d(C_1, \dots, C_s)$ denote the weighted number of (possibly disconnected) d -fold coverings of \mathbb{P}^1 , ramified over s fixed points of \mathbb{P}^1 with monodromies in fixed conjugacy classes C_1, \dots, C_s of the symmetric group S_d .¹³ The generating function for degree d coverings of \mathbb{P}^1 whose ramification over $0, \infty \in \mathbb{P}^1$ can be fixed arbitrarily while the remaining $b = s - 2$ ramifications are simple is given by

$$(4.17) \quad Z_{\text{Hur}}^{(2)}(x, y, q, Q) = \sum_{d, b=0}^{\infty} Q^d \frac{(-g_s)^b}{b!} \sum_{|\mu|=|\nu|=d} x_{\mu} y_{\nu} H_d(C_{\mu}, C_{\nu}, \underbrace{C_{(2)}, \dots, C_{(2)}}_{b \text{ times}}),$$

where C_{μ} is the conjugacy class corresponding to the partition μ of d , $C_{(2)}$ is the conjugacy class of a transposition, and $x_{\mu} = \prod_i x_{\mu_i}$ for a set of variables $x = (x_1, x_2, \dots)$. The generating function for connected coverings is the corresponding free energy

$$(4.18) \quad F_{\text{Hur}}^{(2)}(x, y, v, q) = \log Z_{\text{Hur}}^{(2)}(x, y, v, q) = \sum_{d, b=0}^{\infty} Q^d \frac{(-g_s)^b}{b!} \sum_{|\mu|=|\nu|=d} x_{\mu} y_{\nu} H_{d,b}^{\bullet}(\mu, \nu),$$

where $H_{d,b}^{\bullet}(\mu, \nu)$ denote the double Hurwitz numbers introduced by Okounkov which are the weighted numbers of connected degree d coverings of \mathbb{P}^1 with monodromy around $0, \infty \in \mathbb{P}^1$ given by μ and ν , respectively, and b are the numbers of additional simple ramifications.

The difference between simple Hurwitz numbers and double Hurwitz numbers is as follows. Let μ be a partition of an integer d . Then the simple Hurwitz number h_{μ}^g is the number of connected genus g branched covers of \mathbb{P}^1 with monodromy given by μ over a fixed point, usually identified with ∞ , and an appropriate number of fixed simple branch points. In the more general setting of double Hurwitz numbers, there is a useful explicit expression for the partition function (4.17) given by

$$(4.19) \quad Z_{\text{Hur}}^{(2)}(x, y, v, q) = \sum_{\lambda} Q^{|\lambda|} q^{\kappa(\lambda)/2} \mathfrak{s}_{\lambda}(x) \mathfrak{s}_{\lambda}(y),$$

which is proven in [34] to be a tau-function of the Toda lattice hierarchy. Here $\kappa(\lambda)$ is one of the shifted symmetric polynomials that has the explicit expression

$$(4.20) \quad \kappa(\lambda) = \sum_i \lambda_i (\lambda_i - 2i + 1),$$

which is essentially the Casimir eigenvalue $C_2(\lambda)$ [96]. It is then clear that (4.19) can be specified to describe (4.15) simply by choosing $x = y = Q^{-1/2}(1, \dots, 1)$. This implies that the Yang-Mills partition function $Z_{\text{YM}}^{U(N)}(\mathbb{P}^1)$ is a tau-function of the Toda lattice hierarchy.

Once again, the q -deformed case is the one more directly connected to Donaldson-Thomas theory, and the formula (4.19) applies to that case as well via a suitable specification of $x =$

¹³The genus g of the covering surface is determined in terms of the branching structure and the degree d by the Riemann-Hurwitz formula.

(x_1, x_2, \dots) and $y = (y_1, y_2, \dots)$. The heat kernel expansion for the partition function of q -deformed two-dimensional Yang-Mills theory on the sphere can be written in terms of Schur polynomials as [97]

$$(4.21) \quad Z_{q\text{-YM}}^{U(N)}(\mathbb{P}^1) = \sum_{\lambda} \mathfrak{s}_{\lambda}(1, q, \dots, q^{N-1})^2 q^{p C_2(\lambda)} ,$$

where $\mathfrak{s}_{\lambda}(1, q, \dots, q^{N-1}) = \dim_q \lambda$ is the quantum dimension of the $U(N)$ representation associated to λ . As above, it can be interpreted as a tau-function of the Toda lattice hierarchy and it is also a generating function for double Hurwitz numbers. As before, due to the argument of the quadratic versus linear behaviour of the exponential term in (4.21), it is the partition function of the $U(\infty)$ chiral sector of q -deformed Yang-Mills theory that can be related to the orbifold partition function $\tilde{Z}_{\text{DT}}^{\mathbb{Y}}(q^p, Q)$ (or to the instanton partition function of the K-theory version of the gauge theory above).

4.5. Equivariant Gromov-Witten theory. According to the work of Gross and Taylor [98, 99, 100], two-dimensional Yang-Mills theory has a large N string expansion based on branched coverings of \mathbb{P}^1 . The Hurwitz theory involved is the one that enumerates simple Hurwitz numbers [99, 100, 101]. In the formalism of double Hurwitz numbers, it corresponds to taking $x_2 = x_3 = \dots = 0$ above. On the other hand, in the application of [34] to two-dimensional Yang-Mills theory and its q -deformation on \mathbb{P}^1 the two sets of variables $x = y$ are taken to be equal, in order to describe the \mathfrak{s}_{λ}^2 terms of the partition functions (as occurs throughout this paper). Hence only a particular case of (4.19) is required, and this seems to indicate that it is more natural to consider two branch points at 0 and ∞ , instead of just one, with the same monodromies. This is true for both the ordinary and the q -deformed case, changing only the branching data which depends on the parameter q in the latter case.

The q -deformed gauge theory conjecturally describes a non-perturbative completion of the A-model topological string theory on the local toric Calabi-Yau threefold given by the total space of the rank two holomorphic bundle $\mathbb{X}_p = \mathcal{O}_{\mathbb{P}^1}(p-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-p)$ over \mathbb{P}^1 (generalizing the resolved conifold $\mathbb{X}_1 = \mathbb{X}$ and the $\mathbb{C}^3/\mathbb{Z}_2$ orbifold resolution $\mathbb{X}_2 = \mathbb{Y}$). The closed perturbative topological string partition function on \mathbb{X}_p is given by [97, 101]

$$(4.22) \quad Z_{\text{top}}^{\mathbb{X}_p}(q, Q) = \sum_{\lambda} (-1)^{p|\lambda|} Q^{|\lambda|} q^{(p-2)\kappa(\lambda)/2} W_{\lambda}(q)^2 ,$$

where

$$(4.23) \quad W_{\lambda}(q) = q^{-\kappa(\lambda)/4} \prod_{u \in \lambda} \frac{1}{[h(u)]}$$

is a specialization of the topological vertex [6]. In terms of the principal q -specialization of the Schur functions given by (2.7), the topological string partition function reads

$$(4.24) \quad Z_{\text{top}}^{\mathbb{X}_p}(q, Q) = \sum_{\lambda} (-1)^{p|\lambda|} Q^{|\lambda|} q^{-2n(\lambda)+(p-3)\kappa(\lambda)/2} \mathfrak{s}_{\lambda}(q^{i-1/2})^2 ,$$

where we have dropped an irrelevant overall constant.

Two distinct simplifications of (4.24) occur for $p = 1$

$$(4.25) \quad Z_{\text{top}}^{\mathbb{X}_1}(q, Q) = \sum_{\lambda} (-Q)^{|\lambda|} q^{-\sum_i \lambda_i^2 + |\lambda|} \mathfrak{s}_{\lambda}(q^{i-1/2})^2 ,$$

and for $p = 3$

$$(4.26) \quad Z_{\text{top}}^{\mathbb{X}_3}(q, Q) = \sum_{\lambda} (-Q)^{|\lambda|} q^{-2n(\lambda)} \mathfrak{s}_{\lambda}(q^{i-1/2})^2 .$$

As explained in [101], here $Z_{\text{top}}^{\mathbb{X}_p}(q, Q)$ is an equivariant partition function which describes a topological string theory that is generically different from the standard one. For $p = 1$ it gives the usual topological string theory on the resolved conifold \mathbb{X} , and corresponds to a 1-Toda tau-function. For $p = 2$ the equivariant topological string theory has partition function $Z_{\text{top}}^{\mathbb{X}_2}(q, Q) = Z_{\text{top}}^{\mathbb{X}_1}(q, Q)^{-1}$, consistently with the duality (2.24) between the corresponding Donaldson-Thomas theories. The case $p = 3$ corresponds to the local \mathbb{P}^2 partition function on $\mathcal{O}_{\mathbb{P}^2}(-3) \rightarrow \mathbb{P}^2$ [101]. In this instance there is no second Casimir term and hence no quadratic dependence on the boxes of the representation λ_i . This seems to imply that it corresponds to a 2-Toda tau-function, in contrast to its non-perturbative completion, which is presumably related to the fact that the local \mathbb{P}^2 geometry contains a compact four-cycle. It would be interesting to better understand in general which topological string theories correspond to 2-Toda hierarchies, for which the string theory may be more naturally described using Okounkov's formalism of double Hurwitz numbers.

5. VICIOUS WALKERS AND STOCHASTIC GROWTH REPRESENTATION

5.1. Lock-step model. In [45], we find an interpretation of certain random unitary matrix model averages

$$(5.1) \quad \left\langle \prod_{i=1}^N \det(I + x_i U^\dagger) \det(I + y_i U) \right\rangle_{U(N)} \\ = \prod_{j=1}^N \int_0^{2\pi} \frac{d\theta_j}{2\pi} \prod_{i=1}^N \left(1 + x_i e^{-i\theta_j}\right) \left(1 + y_i e^{i\theta_j}\right) \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^2$$

in terms of configurations of weighted non-intersecting lattice paths. The non-intersecting path model associated to (5.1) is the lock-step model, with a slight variation. It is explained in detail in [45] and we summarize their description in what follows. On the x -axis, the allowed points are $x = 1, 2, \dots, N$. The procedure is that each point is moved to the line $y = 1$, according to the rule that each x coordinate must either stay the same (unit weight) or increase by one (weight x_1), always with all x coordinates remaining distinct. This procedure is repeated a total of N times, with each right diagonal segment at step j weighted by x_j as x coordinates are moved to the line $y = j$. Notice the difference with the original lock-step model, where at each time all particles move either to their right or to their left with equal probability. After step N , perform another N steps, but now with the segments either vertical (unit weight) or left diagonal (weight y_{2N+1-j} at step $N + j$). As usual, we have the conditions that the segments do not intersect and, in addition, are further constrained to return at the line $y = 2N$ to the same initial x coordinates.

We thus have, as was the case in the Brownian motion description of Chern-Simons partition functions [40], a reunion condition on the walkers. The result of [45] is that the generating function for this process is given by (5.1). As we have seen in Section 3.2, in the limit $N \rightarrow \infty$ with the weights of the process taken to be $x_i = y_i = q^{i-1/2}$, this is the Donaldson-Thomas partition function $Z_{\text{DT}}^{\mathbb{C}^3}(q)$. Since $N = \infty$ there are infinitely many walkers. Likewise, the reduced orbifold Donaldson-Thomas partition function $\tilde{Z}_{\text{DT}}^{\mathbb{Y}}(q, Q)$ is the generating function for this process with weights $x_i = y_i = \sqrt{Q}(-q)^{i-1/2}$. More general geometries correspond to higher-dimensional versions of this process, with the walkers constrained to move independently on coordinate planes. In this picture, wall-crossing phenomena appear as the creation or destruction of coordinate planes, and hence particles, and changes in the weightings of walkers.

The celebrated Robinson-Schensted-Knuth correspondence [49] establishes a bijection between weighted $N \times N$ non-negative integer matrices with entries a_{ij} weighted by $(x_i y_j)^{a_{ij}}$ and pairs of weighted semi-standard tableaux of content N . This implies that (2.2) is the generating function for the weighted matrices, and hence the MacMahon function $M(q)$ is the generating function for infinite-dimensional matrices with entries a_{ij} weighted by $(q^{(i-1/2)(j-1/2)})^{a_{ij}}$. Moreover, in [45] it is found that the function

$$(5.2) \quad \prod_{i,j=1}^N (1 - x_i x_j)^{-2}$$

is the generating function for $2N \times 2N$ matrices $[a_{ij}]$ which are invariant under reflections about the entry $(N + \frac{1}{2}, N + \frac{1}{2})$, i.e. $a_{ij} = a_{2N+1-i, 2N+1-j}$. If we specify $x_i = q^{i-1/2}$ and take the limit $N \rightarrow \infty$, the generating function is the square of the MacMahon function, a factor that appears in the Donaldson-Thomas partition functions of the conifold and the $\mathbb{C}^3/\mathbb{Z}_2$ orbifold.

Interestingly enough, instead of considering a symmetry constraint on the matrices, one can impose constraints on the entries. In particular, if we constrain them to be 0 or 1, with the entries a_{ij} weighted by $(x_i y_j)^{a_{ij}}$, then the generating function is the right-hand side of the dual Cauchy identity (2.19) [45]. Recall that this leads to the Schur expansion (2.20) of the Donaldson-Thomas partition function on the resolved conifold. Thus the reduced partition function $\tilde{Z}_{\text{DT}}^{\times}(q, Q)$ can be interpreted as the generating function of such matrices, with weights $x_i = -\sqrt{Q}(-q)^{i-1/2}$ and $y_j = -\sqrt{Q}(-q)^{j-1/2}$. These alternative enumerative descriptions in terms of infinite non-negative integer matrices are intriguing, in light of the fact that in general the Donaldson-Thomas partition function Z_{DT}^X of a threefold X is a generating function which counts ideal sheaves on X . It would be interesting to formulate a more direct connection between these infinite-dimensional random matrix theories and the geometric counting problems.

5.2. Corner growth model. It is also possible to relate the Donaldson-Thomas partition functions to the distributional limit of the corner growth (or last passage) model with geometric weights [18]. Let $\omega(i, j)$, $(i, j) \in \mathbb{Z}^2$ be independent geometric random variables and define

$$(5.3) \quad G(M, N) = \max_{\pi} \sum_{(i,j) \in \pi} \omega(i, j) ,$$

where the maximum is taken over all up/right paths π from $(1, 1)$ to (M, N) ; this is called the corner growth model [18]. It is shown in [19] that if $\omega(i, j)$ are independent geometric random variables with probability distribution of the form

$$(5.4) \quad \mathcal{P}[\omega(j, k) = m] = (1 - x_j y_k) (x_j y_k)^m ,$$

then the probability that $G(M, N)$ is smaller than a certain value can be written in terms of the Schur measure as

$$(5.5) \quad \mathcal{P}[G(M, N) \leq t] = \left(\prod_{j,k=1}^n (1 - x_j y_k) \right) \sum_{\lambda: \lambda_1 \leq t} \mathfrak{s}_{\lambda}(x) \mathfrak{s}_{\lambda}(y) .$$

Since it is given in terms of the Schur measure, the choice of parameters $x_j = q^{j-1/2}$ and $y_k = q^{k-1/2}$ makes the normalization constant of the process in the limit $n \rightarrow \infty$ (the normalization constant for $t \rightarrow \infty$) equal to the Donaldson-Thomas partition function for \mathbb{C}^3 in (2.1).

As in the case of the lock-step model, since the expression (5.5) is very general as it involves generic coefficients x_j and y_k , it also leads to the orbifold partition function (2.25) by again choosing $x_j = \sqrt{Q}(-q)^{j-1/2}$ and $y_k = \sqrt{Q}(-q)^{k-1/2}$. To describe the conifold partition

function (2.20), we need to construct the dual process and hence modify (5.4) to

$$(5.6) \quad \mathcal{P}^\vee[\omega(j, k) = m] = (1 + x_j y_k)^{-1} \Xi_b(x_j y_k)^m,$$

where Ξ_b is the endomorphism of the ring of symmetric polynomials given by $\Xi_b(\epsilon_\lambda) = \mathfrak{h}_\lambda$ for every Young diagram λ , with ϵ_λ the elementary symmetric polynomials and \mathfrak{h}_λ the homogeneous symmetric polynomials [49]. Then

$$(5.7) \quad \mathcal{P}^\vee[G(M, N) \leq t] = \left(\prod_{j,k=1}^n (1 + x_j y_k)^{-1} \right) \sum_{\lambda: \lambda_1 \leq t} \mathfrak{s}_\lambda(x) \mathfrak{s}_{\lambda'}(y),$$

and the choice $x_j = -\sqrt{Q}(-q)^{j-1/2}$ and $y_k = -\sqrt{Q}(-q)^{k-1/2}$ makes the normalization constant of the process equal to the partition function (2.20) in the limit $n \rightarrow \infty$. More general geometries correspond to multiple copies of this process involving independent random variables. Wall-crossing in this picture is the creation or destruction of independent random variables and changes of weightings.

A particular case of this stochastic growth model can be understood as a generalization of the stochastic process underlying the longest increasing subsequence problem. The distribution of the Poissonized version of the random variable $L(\alpha)$ describing the longest increasing subsequence in a random permutation is given by the Gross-Witten model [102]. The choice of the distribution (5.4) is a generalization of the original model introduced in [19] which has $x_j = x$, $y_k = 1$ for all j, k . If one takes $x = \alpha/N^2$, then $G(N, N)$ converges in distribution to $L(\alpha)$ as $N \rightarrow \infty$, and so one can view $G(N, N)$ as a generalization of the random variable $L(\alpha)$.

This model is intimately related to other growth models, as discussed in detail in [18, 19]. A growth model is a stochastic evolution for a height function $h(x, t)$, with x denoting space and t denoting time. An admissible height function has to satisfy $h(x+1, t) - h(x, t) = \pm 1$ for all t . In particular, the discrete polynuclear growth model is a local random growth model defined inductively by

$$(5.8) \quad h(x, t+1) = \max(h(x-1, t), h(x, t), h(x+1, t)) + \omega(x, t+1)$$

with $(x, t) \in \mathbb{Z} \times \mathbb{N}$ and $h(x, 0) = 0$, where $\omega(x, t)$ are independent random variables. One can think of $h(x, t)$ as the height above x at time t , so that the map $x \mapsto h(x, t)$ describes an interface evolving in time. The special case where $\omega(x, t) = 0$ if $t - x$ is even or if $|x| > t$, and

$$(5.9) \quad w(i, j) = \omega(i-j, i+j-1)$$

for $(i, j) \in \mathbb{Z}^2$ are independent geometric random variables with probability distribution (5.4), yields $G(i, j) = h(i-j, i+j-1)$. This growth model is expected to fall in the Kardar-Parisi-Zhang universality class.

It is also possible to demonstrate the equivalence with other systems, such as random tilings of Aztec diamonds (which is related to a dimer model), and non-intersecting walks on a graph [18, 19]. The latter correspondence, in the case of the distribution (5.4) that leads to Donaldson-Thomas theory, is known in detail [15] but its description is rather lengthy. It seems that the lock-step model [45] described above should be directly related to this version of the corner growth model since, as we have seen above, the probability (5.5) without the normalization is the generating function of that vicious walkers model.

REFERENCES

- [1] S. K. Donaldson and R. P. Thomas, ‘‘Gauge theory in higher dimensions,’’ in: *The Geometric Universe: Science, Geometry, and the Work of Roger Penrose*, S. Huggett et al. eds., Oxford University Press (1998) 31–47.

- [2] R. P. Thomas, “A holomorphic Casson invariant for Calabi-Yau 3-folds and bundles on $K3$ fibrations,” *J. Diff. Geom.* **54** (2000) 367–438 [arXiv:math.AG/9806111].
- [3] A. Iqbal, N. A. Nekrasov, A. Okounkov and C. Vafa, “Quantum foam and topological strings,” *J. High Energy Phys.* **0804** (2008) 011 [arXiv:hep-th/0312022].
- [4] M. Cirafici, A. Sinkovics and R. J. Szabo, “Cohomological gauge theory, quiver matrix models and Donaldson-Thomas theory,” *Nucl. Phys. B* **809** (2009) 452–518 [arXiv:0803.4188 [hep-th]].
- [5] A. Okounkov, N. Reshetikhin and C. Vafa, “Quantum Calabi-Yau and classical crystals,” *Progr. Math.* **244** (2006) 597–618 [arXiv:hep-th/0309208].
- [6] M. Aganagic, A. Klemm, M. Marino and C. Vafa, “The topological vertex,” *Commun. Math. Phys.* **254** (2005) 425–478 [arXiv:hep-th/0305132].
- [7] D. Maulik, N. A. Nekrasov, A. Okounkov and R. Pandharipande, “Gromov-Witten theory and Donaldson-Thomas theory I,” *Compos. Math.* **142** (2006) 1263–1285 [arXiv:math.AG/0312059].
- [8] D. Maulik, A. Oblomkov, A. Okounkov and R. Pandharipande, “Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds,” [arXiv:0809.3976 [math.AG]].
- [9] S. V. Kerov, *Asymptotic Representation Theory of the Symmetric Group and its Applications in Analysis* (American Mathematical Society, 2003).
- [10] N. A. Nekrasov and A. Okounkov, “Seiberg-Witten theory and random partitions,” *Progr. Math.* **244** (2006) 525–596 [arXiv:hep-th/0306238].
- [11] A. S. Losev, A. Marshakov and N. A. Nekrasov, “Small instantons, little strings and free fermions,” in: *From Fields to Strings: Circumnavigating Theoretical Physics*, Vol. 1, M. Shifman, A. Vainshtein and J. F. Wheeler eds., World Scientific (2004) 581–621 [arXiv:hep-th/0302191].
- [12] A. Okounkov, “Infinite wedge and random partitions,” *Selecta Math. (N.S.)* **7** (2001) 57–81 [arXiv:math/9907127].
- [13] J. Fulman, “ $GL(n, q)$ and increasing subsequences in non-uniform random permutations,” *Ann. Combin.* **6** (2002) 19–32 [arXiv:math/0109079].
- [14] E. Strahov, “A differential model for the deformation of the Plancherel growth process,” *Adv. Math.* **217** (2008) 2625–2663 [arXiv:0706.3292 [math.CO]].
- [15] K. Johansson, “Random matrices and determinantal processes,” in: *Mathematical Statistical Physics*, A. Bovier, F. Dunlop, A. van Enter, F. den Hollander and J. Dalibard eds., Elsevier Science (2006) 1–56 [arXiv:math-ph/0510038].
- [16] J. Ben Hough, M. Krishnapur, Y. Peres and B. Virág, “Determinantal processes and independence,” *Prob. Surveys* **3** (2006) 206–229 [arXiv:math/0503110].
- [17] A. Borodin, “Determinantal point processes,” [arXiv:0911.1153 [math.PR]].
- [18] K. Johansson, “Shape fluctuations and random matrices,” *Commun. Math. Phys.* **209** (2000) 437–476 [arXiv:math/9903134].
- [19] K. Johansson, “Non-intersecting paths, random tilings and random matrices,” *Prob. Theory Relat. Fields* **123** (2002) 225–280 [arXiv:math/0011250].
- [20] I. G. Macdonald, *Symmetric Functions and Hall Polynomials* (Oxford University Press, 1979).
- [21] R. J. Szabo, “Instantons, topological strings and enumerative geometry,” arXiv:0912.1509 [hep-th].
- [22] T. Maeda, T. Nakatsu, K. Takasaki and T. Tamakoshi, “Five-dimensional supersymmetric Yang-Mills theories and random plane partitions,” *J. High Energy Phys.* **0503** (2005) 056 [arXiv:hep-th/0412327].
- [23] T. Nakatsu and K. Takasaki, “Melting crystal, quantum torus and Toda hierarchy,” *Commun. Math. Phys.* **285** (2009) 445–468 [arXiv:0710.5339 [hep-th]].
- [24] P. Sulkowski, “Deformed boson-fermion correspondence, Q -bosons, and topological strings on the conifold,” *J. High Energy Phys.* **0810** (2008) 104 [arXiv:0808.2327 [hep-th]].
- [25] I. M. Gessel, “Symmetric functions and P-recursiveness,” *J. Combin. Theory A* **53** (1990) 257–285.
- [26] G. Szegő, “Ein Grenzwertsatz über die toeplitzschen determinanten, einer reellen positiven funktion,” *Math. Ann.* **76** (1915) 490–503.
- [27] B. Eynard, “A matrix model for plane partitions and TASEP,” *J. Stat. Mech.* **0910** (2009) P10011 [arXiv:0905.0535 [math-ph]].
- [28] B. Eynard, A. K. Kashani-Poor and O. Marchal, “A matrix model for the topological string I: Deriving the matrix model,” [arXiv:1003.1737 [hep-th]].
- [29] T. Okuda, “Derivation of Calabi-Yau crystals from Chern-Simons gauge theory,” *J. High Energy Phys.* **0503** (2005) 047 [arXiv:hep-th/0409270].
- [30] M. Tierz, “Soft matrix models and Chern-Simons partition functions,” *Mod. Phys. Lett. A* **19** (2004) 1365–1378 [arXiv:hep-th/0212128].

- [31] R. J. Szabo and M. Tierz, “Chern-Simons matrix models, two-dimensional Yang-Mills theory and the Sutherland model,” arXiv:1003.1228 [hep-th].
- [32] M. Sato and Y. Sato, “Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold,” Lect. Notes Num. Appl. Anal. **5** (1982) 259–271.
- [33] G. Segal and G. Wilson, “Loop groups and equations of KdV type,” Publ. Math. IHES **91** (1985) 5–65.
- [34] A. Okounkov, “Toda equations for Hurwitz numbers,” Math. Res. Lett. **7** (2000) 447–453 [arXiv:math/0004128].
- [35] S. K. Lando, “Ramified coverings of the two-dimensional sphere and the intersection theory in spaces of meromorphic functions on algebraic curves,” Russian Math. Surveys **57** (2002) 463–533.
- [36] J. Zhou, “Hodge integrals and integrable hierarchies,” [arXiv:math.AG/0310408].
- [37] T. Eguchi, K. Hori and S. K. Yang, “Topological sigma models and large N matrix integral,” Int. J. Mod. Phys. A **10** (1995) 4203–4224 [arXiv:hep-th/9503017].
- [38] A. Okounkov and R. Pandharipande, “The equivariant Gromov-Witten theory of \mathbb{P}^1 ,” Ann. Math. **163** (2006) 561–605 [arXiv:math.AG/0207233].
- [39] S. Karlin and J. McGregor, “Coincidence probabilities,” Pacific J. Math. **9** (1959) 1141–1164.
- [40] S. de Haro and M. Tierz, “Brownian motion, Chern-Simons theory, and 2D Yang-Mills,” Phys.Lett. B **601** (2004) 201–208 [arXiv:hep-th/0406093].
- [41] S. de Haro, “Chern-Simons theory, 2D Yang-Mills, and Lie algebra wanderers,” Nucl. Phys. B **730** (2005) 312–351 [arXiv:math/0412110].
- [42] P. G. de Gennes, “A soluble model for fibrous structures with steric constraints,” J. Chem. Phys. **48** (1968) 2257–2259.
- [43] M. E. Fisher, “Walks, walls, wetting, and melting,” J. Stat. Phys. **34** (1984) 667–729.
- [44] D. A. Huse and M. E. Fisher, “Commensurate melting, domain walls, and dislocation,” Phys. Rev. B **29** (1984) 239–270.
- [45] P. J. Forrester and A. Gamburd, “Counting formulas associated with some random matrix averages,” J. Combin. Theory A **113** (2005) 934–951 [arXiv:math/0503002].
- [46] M. Kardar, G. Parisi and Y. C. Zhang, “Dynamic Scaling of Growing Interfaces,” Phys. Rev. Lett. **58** (1986) 889–892.
- [47] H. Ooguri, P. Sulkowski and M. Yamazaki, “Wall crossing as seen by matrix models,” [arXiv:1005.1293 [hep-th]].
- [48] P. A. MacMahon, *Combinatory Analysis I–II* (Chelsea, 1960).
- [49] R. P. Stanley, *Enumerative Combinatorics. Volume 2* (Cambridge University Press, 2001).
- [50] J. Fuchs, *Affine Lie Algebras and Quantum Groups: An Introduction, with Applications in Conformal Field Theory* (Cambridge University Press, 1995).
- [51] B. Szendrői, “Noncommutative Donaldson-Thomas theory and the conifold,” Geom. Topol. **12** (2008) 1171–1202 [arXiv:0705.3419 [math.AG]].
- [52] B. Young and J. Bryan, “Generating functions for coloured 3D Young diagrams and the Donaldson-Thomas invariants of orbifolds,” Duke Math. J. **152** (2010) 115–153 [arXiv:0802.3948 [math.CO]].
- [53] D. Joyce and Y. Song, “A theory of generalized Donaldson-Thomas invariants,” [arXiv:0810.5645 [math.AG]].
- [54] M. Aganagic, H. Ooguri, C. Vafa and M. Yamazaki, “Wall crossing and M-theory,” [arXiv:0908.1194 [hep-th]].
- [55] S. Cecotti and C. Vafa, “BPS wall crossing and topological strings,” arXiv:0910.2615 [hep-th].
- [56] D. L. Jafferis and G. W. Moore, “Wall crossing in local Calabi-Yau manifolds,” [arXiv:0810.4909 [hep-th]].
- [57] W. y. Chuang and D. L. Jafferis, “Wall crossing of BPS states on the conifold from Seiberg duality and pyramid partitions,” Commun. Math. Phys. **292** (2009) 285–301 [arXiv:0810.5072 [hep-th]].
- [58] R. P. Stanley, “Some combinatorial properties of Jack symmetric functions,” Adv. Math. **77** (1989) 76–115.
- [59] A. Gholampour and Y. Jiang, “Counting invariants for the ADE McKay quivers,” [arXiv:0910.5551 [math.AG]].
- [60] A. Iqbal and A. K. Kashani-Poor, “The vertex on a strip,” Adv. Theor. Math. Phys. **10** (2006) 317–343 [arXiv:hep-th/0410174].
- [61] K. Nagao, “Derived categories of small toric Calabi-Yau 3-folds and counting invariants,” [arXiv:0809.2994 [math.AG]].
- [62] T. J. Hollowood, A. Iqbal and C. Vafa, “Matrix models, geometric engineering and elliptic genera,” J. High Energy Phys. **0803** (2008) 069 [arXiv:hep-th/0310272].

- [63] N. Caporaso, M. Cirafici, L. Griguolo, S. Pasquetti, D. Seminara and R. J. Szabo, “Topological strings, two-dimensional Yang-Mills theory and Chern-Simons theory on torus bundles,” *Adv. Theor. Math. Phys.* **12** (2008) 981–1058 [arXiv:hep-th/0609129].
- [64] L. Onsager, “Crystal statistics. I. A two-dimensional model with an order-disorder transition,” *Phys. Rev.* **65** (1944) 117–149.
- [65] E. W. Montroll, R. B. Potts, and J. C. Ward, “Correlations and spontaneous magnetization of the two-dimensional Ising model,” *J. Math. Phys.* **4** (1963) 308–322.
- [66] I. Bars and F. Green, “Complete integration of $U(N)$ lattice gauge theory in a large N limit,” *Phys. Rev. D* **20** (1979) 3311–3330.
- [67] D. J. Gross and E. Witten, “Possible third order phase transition in the large N lattice gauge theory,” *Phys. Rev. D* **21** (1980) 446–453.
- [68] D. S. Lubinsky and E. B. Saff, “Convergence of Padé approximants of partial theta-functions and the Rogers-Szegő polynomials,” *Constr. Approx.* **3** (1987) 331–361.
- [69] C. A. Tracy and H. Widom, “On the distributions of the lengths of the longest monotone subsequences in random words,” *Prob. Theory Relat. Fields* **119** (2001) 350–380 [arXiv:math/9904042].
- [70] R. Askey, “Orthogonal polynomials and theta-functions,” *Proc. Symp. Pure Math.* **49** (1989) 299–321.
- [71] C. Vafa and E. Witten, “A strong coupling test of S-duality,” *Nucl. Phys. B* **431** (1994) 3–77 [arXiv:hep-th/9408074].
- [72] G. Bonelli and A. Tanzini, “Topological gauge theories on local spaces and black hole entropy countings,” *Adv. Theor. Math. Phys.* **12** (2008) 1429–1446 [arXiv:0706.2633 [hep-th]].
- [73] L. Griguolo, D. Seminara, R. J. Szabo and A. Tanzini, “Black holes, instanton counting on toric singularities and q -deformed two-dimensional Yang-Mills theory,” *Nucl. Phys. B* **772** (2007) 1–24 [arXiv:hep-th/0610155].
- [74] R. M. Gray, “Toeplitz and circulant matrices: A review,” *Found. Trends Commun. Inform. Theory* **2** (2006) 155–239.
- [75] J. S. Geronimo and K. M. Case, “Scattering theory and polynomials orthogonal on the unit circle,” *J. Math. Phys.* **20** (1979) 299–310.
- [76] A. Borodin and A. Okounkov, “A Fredholm determinant formula for Toeplitz determinants,” *Integral Equations Operator Theory* **37** (2000) 386–396 [arXiv:math.CA/9907165].
- [77] J. Baik and E. M. Rains, “Algebraic aspects of increasing subsequences,” *Duke Math. J.* **109** (2001) 1–65 [arXiv:math/9905083].
- [78] G. W. Moore, “Geometry of the string equations,” *Commun. Math. Phys.* **133** (1990) 261–304.
- [79] V. A. Kazakov, I. K. Kostov and D. Kutasov, “A matrix model for the two-dimensional black hole,” *Nucl. Phys. B* **622** (2002) 141–188 [arXiv:hep-th/0101011].
- [80] A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov and A. Orlov, “Matrix models of $2D$ gravity and Toda theory,” *Nucl. Phys. B* **357** (1991) 565–618.
- [81] I. M. Krichever, “The tau-function of the universal Whitham hierarchy, matrix models and topological field theories,” *Commun. Pure Appl. Math.* **47** (1994) 437–475 [arXiv:hep-th/9205110].
- [82] M. Adler and P. van Moerbeke, “Integrals over classical groups, random permutations, Toda and Toeplitz lattices,” *Commun. Pure Appl. Math.* **54** (2001) 153–205 [arXiv:math/9912143].
- [83] K. Ueno and K. Takasaki, “Toda lattice hierarchy,” *Adv. Studies Pure Math.* **4** (1984) 1–95.
- [84] M. Adler and P. van Moerbeke, “Recursion relations for unitary integrals, combinatorics and the Toeplitz lattice,” *Commun. Math. Phys.* **237** (2003) 397–440 [arXiv:math-ph/0201063].
- [85] M. J. Ablowitz and J. F. Ladik, “Nonlinear differential-difference equations,” *J. Math. Phys.* **16** (1975) 598–603.
- [86] M. Cafasso, “Matrix biorthogonal polynomials on the unit circle and non-abelian Ablowitz-Ladik hierarchy,” *J. Phys. A* **42** (2009) 365211 [arXiv:0804.3572 [math.CA]].
- [87] J. Harnad and A. Yu. Orlov, “Convolution symmetries of integrable hierarchies, matrix models and tau-functions,” [arXiv:0901.0323 [math-ph]].
- [88] A. R. Its, C. A. Tracy and H. Widom, “Random words, Toeplitz determinants and integrable systems. II,” *Physica D* **152–153** (2001) 199–224 [arXiv:nlin/0004018].
- [89] M. Jimbo, T. Miwa and K. Ueno, “Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I,” *Physica D* **2** (1981) 306–352.
- [90] J. Palmer, “Zeroes of the Jimbo-Miwa-Ueno tau-function,” *J. Math. Phys.* **40** (1999) 6638–6681 [arXiv:solv-int/9810004].
- [91] N. A. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” *Adv. Theor. Math. Phys.* **7** (2004) 831–864 [arXiv:hep-th/0206161].

- [92] T. S. Tai, “Instanton counting and matrix model,” *Progr. Theor. Phys.* **119** (2008) 165–177 [arXiv:0709.0432 [hep-th]].
- [93] A. A. Migdal, “Gauge transitions in gauge and spin lattice systems,” *Sov. Phys. JETP* **42** (1975) 743.
- [94] B. E. Rusakov, “Loop averages and partition functions in $U(N)$ gauge theory on two-dimensional manifolds,” *Mod. Phys. Lett. A* **5** (1990) 693–703.
- [95] D. J. Gross and A. Matytsin, “Instanton induced large N phase transitions in two-dimensional and four-dimensional QCD,” *Nucl. Phys. B* **429** (1994) 50–74 [arXiv:hep-th/9404004].
- [96] A. Okounkov and G. Olshanski, “Shifted Schur functions,” *St. Petersburg Math. J.* **9** (1998) 239–300 [arXiv:q-alg/9605042].
- [97] M. Aganagic, H. Ooguri, N. Saulina and C. Vafa, “Black holes, q -deformed $2D$ Yang-Mills, and non-perturbative topological strings,” *Nucl. Phys. B* **715** (2005) 304–348 [arXiv:hep-th/0411280].
- [98] D. J. Gross, “Two-dimensional QCD as a string theory,” *Nucl. Phys. B* **400** (1993) 161–180 [arXiv:hep-th/9212149].
- [99] D. J. Gross and W. Taylor, “Two-dimensional QCD is a string theory,” *Nucl. Phys. B* **400** (1993) 181–210 [arXiv:hep-th/9301068].
- [100] D. J. Gross and W. Taylor, “Twists and Wilson loops in the string theory of two-dimensional QCD,” *Nucl. Phys. B* **403** (1993) 395–452 [arXiv:hep-th/9303046].
- [101] N. Caporaso, L. Griguolo, M. Marino, S. Pasquetti and D. Seminara, “Phase transitions, double-scaling limit, and topological strings,” *Phys. Rev. D* **75** (2007) 046004 [arXiv:hep-th/0606120].
- [102] J. Baik, P. A. Deift and K. Johansson, “On the distribution of the length of the longest increasing subsequence in a random permutation,” *J. Amer. Math. Soc.* **12** (1999) 1119–1178 [arXiv:math/9810105].

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